THE REPLACEMENT PROBLEM IN FRICTIONAL ECONOMIES: A NEAR-EQUIVALENCE RESULT

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Abstract
We examine how technological change affects wage inequality and unemployment in a calibrated model of matching frictions in the labor market. We distinguish between two polar cases studied in the literature: a "creative destruction" economy, where new machines enter chiefly through new matches and an "upgrading" economy, where machines in existing matches are replaced by new machines. Our main results are: (i) these two economies produce very similar quantitative outcomes, and (ii) the total amount of wage inequality generated by frictions is very small. We explain these findings in light of the fact that, in the model calibrated to the US economy, both unemployment and vacancy durations are very short, i.e., the matching frictions are quantitatively minor. Hence, the equilibrium allocations of the model are remarkably close to those of a frictionless version of our economy where firms are indifferent between upgrading and creative destruction, and where every worker is paid the same market-clearing wage. These results are robust to the inclusion of machine-specific or match-specific heterogeneity into the benchmark model. (JEL: J41, J64, O33)

1. Introduction

Technological progress is gradual and a large part of technological innovations are embodied in capital equipment. New machines tend to be much more productive than existing machines, but typically not every worker is matched with the most advanced equipment. Given that capital-embodied technical change creates substantial productivity differences among machines, what are its effects on wage inequality among labor that uses that equipment? Also, in light of the fact that

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new and more productive capital can make existing capital obsolete, what are the effects of technological change on job creation, job destruction, and, ultimately, unemployment? These are key questions for a macroeconomic analysis of labor markets that takes on a long-run perspective.

If we view the labor market as a frictionless environment where workers are paid according to their marginal product, then the answer to these questions is relatively straightforward and well known. Here, the impact of technological change on inequality is limited to the extent that workers differ in their ability to use capital. Workers who use different vintages of capital but are otherwise identical will be paid the same wage. Moreover, the competitive equilibrium features no unemployment of labor.[1]

On the other hand, if frictions that prevent the free and timeless reallocation of labor among alternative uses are an essential part of the labor market, the situation is far more complex and none of the questions is settled, even in the simple case of homogeneous workers.

First, in frictional models, wages of ex ante equal workers reflect the relative productivity difference of the technology they are working with. That is, wages are not equalized for identical workers due to a "luck" factor: workers are more or less fortunate in the matching process. But how important is this purely frictional source of inequality in terms of measured total wage dispersion for observationally equivalent workers? To put our analysis in perspective, consider that Katz and Autor (1999) report that the 90–10 wage ratio of the residual wage inequality in typical wage regressions, that is, the inequality that remains after controlling for observable characteristics of the workers (gender, race, education, experience) and for individual fixed effects to capture "unobserved individual ability," is around 1.5.

Second, since technological change may require the reallocation of labor towards new and more productive machines, but reallocation requires time with frictions, not all workers will be employed at any time. How does equilibrium unemployment react to changes in the rate of technological change? To this important question, two distinct qualitative answers emerge from the literature.

Aghion and Howitt (1994) argue that when replacing an old obsolete machine with new capital requires destroying the existing match (the Schumpeterian "creative-destruction" model), then unemployment tends to go up as growth accelerates, due to a higher job-separation rate. The models of Caballero and Hammour

1. For the case of worker heterogeneity, there are also precise predictions in the literature. Akerlof (1969) showed that in a frictionless model with heterogeneous labor, unemployment of the worst type of labor can arise in equilibrium only if the aggregate capital stock is fixed and if there is some finite limit to the marginal product of capital. Jovanovic (1998) showed that when capital embodies technological progress and machines are indivisible, faster growth raises wage inequality as long as skills complement capital in the production function. In this case, the most skilled workers will be the ones who are efficiently assigned to work on the most productive machines.

(1998) and Cohen and Saint-Paul (1994) bear similarities to the Aghion–Howitt approach.

In contrast, Mortensen and Pissarides (1998) propose a mechanism whereby the new technologies can replace old machines at some cost, but without destroying the existing match (the costly "upgrading" model).[2] The separation rate is unaffected by faster growth and all the effects work through job creation. For small values of the upgrading cost, unemployment falls with faster growth, thanks to the familiar "capitalization effect": investors are encouraged to create more vacancies, knowing that they will be able to incorporate and benefit from future technological advances.[3]

Though we rely heavily on theory in our analysis—i.e., we explore mechanisms that are made explicit using dynamic general equilibrium theory—our final aim is to offer quantitative answers to these two questions. We wish to evaluate how much frictional wage inequality might be created by technological change, and how different quantitatively the implications for frictional unemployment of the two alternative replacement models really are.

How do we accomplish these tasks? We study a calibrated version of a frictional model à la Diamond–Mortensen–Pissarides (DMP) with random matching and wages determined by cooperative Nash bargaining, augmented with vintage capital à la Solow (1960). The DMP model over the years has established itself as a standard framework of analysis of the labor market (see Pissarides 2000, for an overview of the approach). Since capital-embodied technical change is, arguably, the key driving force of productivity growth in developed economies in the past three decades (see, e.g., Jorgenson 2001), we model technological advancement through the introduction of new capital goods. Technically, our framework is a generalization of Hornstein, Krusell, and Violante (2003) where firms cannot upgrade existing machines without separating from their employed worker.

In particular, we are able to nest the two ways in which new equipment can enter the economy and replace the old one by assuming that (i) new entrant firms can buy new equipment at price I_0 and (ii) existing firms, whether matched or not, can upgrade their equipment to the latest vintage at price I_u. Thus, if $I_u = 0$, technological change is fully match-augmenting and disembodied, whereas if $I_u = \infty$, it has no match-augmenting or disembodied feature at all. In equilibrium, depending on parameter values, new equipment may enter through either channel.

2. In this latter case, but not in the former, one could say that technology is "match-augmenting," because it augments the value of existing matches. In the extreme case where upgrading can proceed at no cost, we have the Solowian model of disembodied technological change, even though the carrier of technology is equipment.

3. An interesting qualification to this result is provided by King and Welling (1995): if, unlike what is customarily assumed in this family of models, workers bear the full fixed-search cost, then the capitalization effect leads to an increase in the number of searchers and to longer unemployment durations.

We parameterize our model economies tightly in order to match some micro-estimates and certain long-run facts. In particular, in a way consistent with our model formulation, we can use data on equipment prices, adjusted for quality, to measure the speed at which the new equipment investment carries technology improvements.[4] In other words, in our framework, the percentage productivity gaps across machines are restricted by data on relative prices of capital.

Our main findings are quite striking. We find that the unemployment effects of a rise in the rate of capital-embodied technological change are, qualitatively, those emphasized by Aghion and Howitt (1994): unemployment rises in response to higher rates of capital-embodied technological change. The difference between the two ways of thinking about how technology is introduced is qualitatively the expected one—the Mortensen–Pissarides, or upgrading, perspective delivers a smaller increase in unemployment than does the Aghion–Howitt perspective—but quantitatively the two models are almost equivalent. Moreover, we find that an acceleration in the speed of technological progress raises wage dispersion, but that the amount of residual wage inequality generated by the calibrated DMP model of technological change is extremely small: the model delivers a 90–10 wage ratio between 1.06 and 1.08, or, at most, one-sixth of that measured.

The intuition for these two results can be found in two key quantitative restrictions that the equilibrium of our calibrated model has to obey. First, the average duration of unemployment in the United States is just above 8 weeks. In search models, the duration of unemployment is proportional to the option value of search and the latter, for risk-neutral workers, depends positively on wage dispersion. A duration of unemployment as short as it is in the data can therefore only be consistent with a very small amount of frictional inequality.[5] Second, the average duration of a vacant job with idle capital in the model (and in the U.S. data) is roughly four weeks, i.e., a very small amount compared to the average life of a new machine, which is over 11 years. When the meeting rate is so high, the firm becomes roughly indifferent between scrapping and upgrading: intuitively, for a given cost of a new machine, the upgrading option is substantially better than the scrapping option only if the matching process is long and costly.

Another way to look at these two conclusions is that, empirically, the labor market frictions are small. When the meeting frictions for both workers and firms disappear, our model converges to the frictionless environment. In the frictionless model: (1) firms are indifferent between upgrading and destroying a job since they face an infinite arrival rate of idle workers, and (2) wage inequality is zero.

4. Hornstein and Krusell (1996), Greenwood, Hercowitz, and Krusell (1997), and Cummins and Violante (2002) provide theoretical justifications and implementations of the price-based approach to the measurement of capital-embodied technical change.

5. Note that in our model, unlike in the more traditional Aghion–Howitt/Mortensen–Pissarides frameworks, there is vacancy heterogeneity in equilibrium, thus unemployed workers face a non-degenerate match distribution upon search.

Incidentally, our results are consistent with the conclusions of Ridder and van den Berg (2003), who estimate the severity of search frictions in the United States, United Kingdom, France, Germany, and the Netherlands and find that the United States has, by a significant amount, the least frictional labor market.

In the second part of the paper, we explore various extensions of the basic framework. The organizing principle we have chosen here is to maintain the assumption that workers are all ex ante identical.[6] Thus, we look at different forms of heterogeneity on the plant level, each of which can influence wages in this form of labor market. There are two cases of interest. First, we consider heterogeneity in equipment/entrepreneurs that goes beyond vintage differences: *machine-specific* heterogeneity. Second, we consider the kind of *match-specific* productivity heterogeneity considered by Mortensen and Pissarides (1994), but now combined with the model of vintage capital. Thus, in the first extension, entrepreneurs that enter the market are diverse and remain diverse as long as they are in the economy, whereas in the second extension, the heterogeneity appears in each new match and remains only so long as each match lasts. Our findings are that these two additional forms of productivity heterogeneity lead to very similar conclusions regarding the replacement debate: in response to a higher rate of capital-embodied technological change unemployment increases, whether or not upgrading is possible, but unemployment increases somewhat less if upgrading is possible. Wage inequality remains limited. Wage inequality is the highest, and becomes particularly visible through the emergence of a group of high-wage earners, in the economy with match-specific heterogeneity, because in this economy the breaking up of a match implies a larger loss of surplus than in an economy with machine-specific heterogeneity.

The outline of the rest of the paper is as follows. In Section 2 we describe the frictionless benchmark of the vintage-capital economy. Sections 3 and 4 contain the two basic versions of the replacement problem in the model with frictions: one where new machines enter in new matches, and one where they mainly enter through the upgrading of machines operated in existing matches. Section 5 contains the calibration to the US economy and quantitative analysis, and also provides more detailed intuition for our two main results. Finally, Section 6 considers extensions to two additional sources of heterogeneity among matches: one due to permanent differences among entrepreneurs/their equipment and one due to permanent differences in match quality. Section 7 concludes.

6. Ex ante worker heterogeneity in productivity levels raises new issues. First, such a model implies wage inequality and unemployment rates that depend nontrivially on worker type, thus making the output of the exercise less comparable to data, since a worker's productivity cannot be measured directly. Second, it is well known (Sattinger 1995) that random matching models with two-sided heterogeneity have multiple steady states. In particular, identical (but identifiable) workers can have different combinations of equilibrium wage and employment rates. These issues are best left for future exploration.

2. The Frictionless Economy

We begin by presenting a version of the Solow (1960) frictionless vintage capital model where production is decentralized into worker-machine pairs operating Leontief technologies: this decentralized production structure is typical in frictional economies. The competitive economy displays neither wage inequality nor unemployment; however, it is a useful benchmark for interpreting the main results of the richer frictional model.

Demographics. Time is continuous. The economy is populated by a stationary measure one of ex ante equal, infinitely lived workers who supply one unit of labor inelastically, and there is also a set of entrepreneurs whose only role it is to pair up with workers and run the firm. Workers and entrepreneurs are risk-neutral and discount the future at rate r.

Production. Production requires pairing one machine and one worker, and machines are characterized by the amount of efficiency units of capital k they embody. A matched worker–machine pair produces a homogeneous output good.

Technological Change. There is embodied and disembodied technical change. The economy wide disembodied productivity level, $e^{z(t)}$, grows at a constant rate $\psi > 0$. Technological progress is also embodied in capital and the amount of efficiency units embodied in new machines grows at the rate $\gamma > 0$. Once capital is installed in a machine it is subject to physical depreciation at the rate $\delta > 0$. A production unit that at time t has age a and is paired with a worker has output

$$y(t, a) = e^{z(t)}k(t, a)^{\omega} = e^{z_0+\psi t}\left[k_0 e^{\gamma(t-a)}e^{-\delta a}\right]^{\omega}, \tag{1}$$

where $\omega > 0$. In what follows we set, without loss of generality, $z_0 = 0$ and $k_0 = 1$.

How New Machines Enter. At any time t firms can freely enter the market upon payment of the initial installation cost $I_0(t)$ for a machine of vintage t. The cost of new vintages grows at the rate g. Existing firms with older machines also have the opportunity to upgrade their machine and bring its productivity up to par with the newest vintage. The cost of upgrading $I_u(t, a) \geq 0$ grows at the rate g but it is independent of age for $a \leq \hat{a}$. We assume that once a machine is too old, it becomes infinitely costly to upgrade: $I_u(t, a) = \infty$ for $a > \hat{a}$.

Rendering the Growth Model Stationary. We will focus on the steady state of the normalized economy; this corresponds to a balanced growth path of the actual economy. It is immediate that for a balanced growth path to exist, we need $g = \psi + \omega\gamma$. In order to make the model stationary we normalize all variables by dividing by the growth factor e^{gt}. The normalized cost of a new production unit is then constant at I_0 and I_u, and the normalized output of a production unit of age a which is paired with a worker is $e^{-\phi a}$, where $\phi \equiv \omega(\gamma + \delta)$; thus, output is defined relative to the newest production unit. Note that the parameter ϕ represents the effective depreciation rate of capital obtained as the sum of physical depreciation δ and technological obsolescence γ.[7]

Embodied Technical Change and the Relative Price of New Capital. Since the cost of new vintage machines in terms of the output good, I_0 and I_u, is growing at the rate g but the number of efficiency units embodied in new vintages is growing at the rate γ, the price of quality-adjusted capital (efficiency units) in terms of output is changing at the rate $g - \gamma = \psi - (1 - \omega)\gamma$. In the quantitative analysis of our model, we will use this relationship to obtain a measure of the rate of embodied technical change from the rate at which the observed relative price of equipment capital changes.

2.1. Competitive Equilibrium

Assume that the labor market is frictionless and competitive so that there is a unique market-clearing wage. In the steady state, the wage rate w, now measured relative to the output of the newest vintage, is constant. Consider a price-taker firm with the newest vintage machine. The firm optimally chooses the age $\bar{a}$ that maximizes the present value of the current machine lifetime profits

$$V_0 = \max_{\bar{a}} \left\{ \int_0^{\bar{a}} e^{-(r-g)a} [e^{-\varphi a} - w] da + e^{-(r-g)\bar{a}} \max\{V_0 - I_u, 0\} \right\}, \quad (2)$$

where $r - g$ is the effective discount factor. While the wage w is the same for all firms, a firm's output falls compared to new machines because of depreciation and obsolescence. Thus, flow profits are monotonically declining in age and eventually become negative, and there is a unique age at which the machine will be discarded or upgraded. Profit maximization leads to the condition

$$(r - g) \max[V_0 - I_u, 0] = e^{-\varphi\bar{a}} - w. \quad (3)$$

The firm upgrades/exits when the gain from upgrading/exit (left-hand side) is the same as the return from a marginal delay of the investment decision (right-hand side). If the value of a new machine is lower (higher) than the cost of upgrading, the firm will scrap (upgrade) the machine. If the firm scraps the machine, the wage is equal to the relative productivity of the oldest machine, which is also the marginal productivity of labor. The higher the wage, the shorter the life-length of capital since (normalized) profits per period fall and thus reach zero sooner.

7. In Hornstein, Krusell, and Violante (2003), we describe the normalization procedure in detail.

Free entry of firms with new machines requires that the value of a new machine does not exceed the cost of a new machine: $V_0 \leq I_0$. Obviously if the cost of upgrading a machine is greater than the cost of buying a new machine, $I_0 < I_u$, then machines will never be upgraded. On the other hand, when the cost of upgrading is lower than the cost of buying a new machine, machines will never exit. Using the profit-maximization condition (3) in the firm value equation (2), the optimal investment condition can be written as

$$\min\{I_0, I_u\} = \int_0^{\bar{a}} e^{-(r-g+\phi)a}\left[1 - e^{-\phi(\bar{a}-a)}\right]da. \tag{4}$$

This is the key condition that determines the exit/upgrading age $\bar{a}$, and hence wages.

Existence and Uniqueness. Equation (4) allows us to discuss existence and uniqueness of the equilibrium. The right-hand side of this equilibrium condition is strictly increasing in the upgrading/exit age $\bar{a}$ for two reasons. First, in an equilibrium with older firms, the relative productivity of the marginal operating firm is lower; therefore, wages have to be lower and profits higher. Second, a longer machine life increases the time span for which profits are accumulated. Define $\bar{r} \equiv r - g + \varphi = r - \psi + \omega\delta$. The right-hand side of (4) increases from 0 to $1/\bar{r}$ as $\bar{a}$ goes from 0 to infinity. Taken together, these facts mean that there exists a unique steady state age $\bar{a}^{CE}$ whenever $min\{I_0, I_u\} < 1/\bar{r}$. This condition is natural: unless you can recover the initial capital investment at zero wages using an infinite lifetime ($\int_0^\infty e^{-\bar{r}a}da = 1/\bar{r}$ being the net profit from such an operation), it is not profitable to start any firm. Finally, with a unit mass of workers, all employed, the firm distribution is uniform with density $1/\bar{a}^{CE}$, which is also the measure of entering/upgrading firms.

The Effect of Growth. Clearly, in this economy neither wage inequality nor unemployment is affected by changes in the rate of technology growth: they are both zero.

3. The Economy with Matching Frictions

Consider now an economy with same preferences, demographics, and technology, but where the labor market is frictional in the sense of Pissarides (2000): an aggregate matching function governs job creation. The nature of the firm's decision process remains the same as in the frictionless economy: there is free entry of firms that buy a new piece of capital, participate in the search process, start producing upon matching with a worker, and finally either upgrade their capital once it becomes too old or exit if upgrading is too costly. Searching is costless: it only takes time. Existing matches dissolve exogenously at the rate σ: upon dissolution, the worker and the firm are thrown into the pool of searchers.[8]

In this environment, vacant firms are heterogeneous in the vintage of their capital for two reasons: first, newly created firms do not match instantaneously, so they remain idle until luck makes them meet a worker; second, firms hit by exogenous separation will also become idle. Note here a key difference with the traditional search-matching framework (Mortensen and Pissarides 1998): the traditional models assume that a new machine can be purchased at no cost and that only posting a vacancy entails a flow cost. This assumption implies that the pool of vacancies consists of the newest machines only, and that only matched machines age over time.[9] Our setup is built on the opposite assumption: purchasing and installing capital is costly—an expense that is sunk when the vacant firm start searching—whereas posting a vacancy is costless.[10] As a result, it can be optimal even for firms with old capital to remain idle.

This class of economic environments is a hybrid between vintage models and matching models. The traditional assumption emphasizes the matching features of the environment, while the explicit distinction between a "large" purchase/setup cost for the machine and a "smaller" search/recruiting flow cost (zero in our model) fits more naturally with its vintage-capital aspects, whose emphasis is on capital investment expenditures as a way of improving productivity. In actual economies, new and old vacancies coexist, as in our setup.

Matching. The number of matches in any moment is determined by a constant returns to scale matching function $m(v, u)$, where $v \equiv \int_0^\infty v(a)da$ is the total number of vacancies of all ages $v(a)$ and u is the total number of unemployed workers. We assume that $m(v, u)$ is strictly increasing in both arguments and satisfies some standard regularity conditions.[11] The rate at which a worker meets a firm with capital of age a is $\lambda_w(a)$ and the rate at which she meets any firm

8. We omitted separations from the description of the competitive equilibrium because, without frictions, it is immaterial whether the match dissolves exogenously or not as the worker can be replaced instantaneously by the firm at no cost.

9. Aghion and Howitt (1994) also describe a vintage capital model with initial setup costs for capital, but they assume that matching is "deterministic": at the time a new machine is set up, a worker queues up for the machine, and after a fixed amount of time the worker and firm start operations. Hence, in the matching process, all vacant firms are equal (although they do not embody the leading-edge technology).

10. Vacancy heterogeneity will survive the addition of a flow search cost, as long as this cost is strictly less than the initial setup cost I_0.

11. In particular,

$$m(0, u) = m(v, 0) = 0,$$
$$\lim_{u\to\infty} m_u(v, u) = \lim_{v\to\infty} m_v(v, u) = 0,$$
$$\lim_{u\to 0} m_u(v, u) = \lim_{v\to 0} m_v(v, u) = +\infty.$$

is $\lambda_w \equiv \int_0^\infty \lambda_w(a)da$, where we assume that the integral is finite. A firm meets a worker at the rate λ_f. Using the notation $\theta = v/u$ to denote labor market tightness, we then have that

$$\lambda_f = \frac{m(\theta, 1)}{\theta}, \tag{5}$$

$$\lambda_w(a) = \frac{v(a)}{v} m(\theta, 1). \tag{6}$$

The expression for the meeting probability in (5) provides a one-to-one (strictly decreasing) mapping between λ_f and θ. Thereafter, when we discuss changes in λ_f, we imagine changes in θ. The measure of worker-firm matches with an a machine is denoted $\mu(a)$ and total employment is $\mu \equiv \int_0^\infty \mu(a)da$.

Scrapping versus Upgrading. Values for the market participants are $J(a)$ and $W(a)$ for matched firms and workers, respectively, $V(a)$ for vacant firms, and U for unemployed workers. We consider two forms of capital replacement and the value functions will differ in the two cases. We proceed immediately by guessing that there is a cutoff age for capital, $\bar{a}$, such that either scrapping or upgrading takes place.

The Values of Matched Workers and Firms. Under both replacement decisions, the flow value of a job for firm and worker for $a \leq \bar{a}$ is given, respectively, by

$$(r-g)J(a) = \max\{e^{-\varphi a} - w(a) - \sigma[J(a) - V(a)] + J'(a), (r-g)V(a)\} \tag{7}$$

$$(r-g)W(a) = \max\{w(a) - \sigma[W(a) - U] + W'(a), (r-g)U\}. \tag{8}$$

The return of a matched firm with an age a machine is equal to profit, i.e., production less wages $w(a)$ paid to the worker, minus the flow rate of capital losses from separation plus the flow losses/gains due to the aging of machines.[12] Analogously for a worker, the return on being in a match with an age a machine is the wage minus the flow rate of capital losses from separation plus the flow losses/gains due to the aging of machines. For the match to be maintained, the flow return from staying in the match must be at least as high as the flow return from leaving the match, i.e., from the firm becoming vacant and the worker becoming unemployed. Note that the capital value equations for matched workers and firms are defined only for matches with machines not older than $\bar{a}$, since all machines either exit or are upgraded at age $\bar{a}$.

12. In Appendix A we describe a typical derivation of the preceding differential equations.

Values of Idle Workers and Firms. In the economy where capital is replaced by scrapping, the values of idle workers and firms are

$$(r-g)V(a) = \max\{\lambda_f[J(a) - V(a)] + V'(a), 0\} \tag{9}$$

$$(r-g)U = b + \int_0^{\bar{a}} \lambda_w(a)[W(a) - U]da. \tag{10}$$

The return on a vacant firm is equal to the capital gain rate from meeting a worker plus the flow losses/gains due to the aging of capital. Vacant machines that are older than the critical age $\bar{a}$ exit.[13] The return for unemployed workers is equal to their benefits b plus the capital gain rate from meeting vacant firms.

When capital is upgraded, we have instead

$$(r-g)V(a) = \begin{cases} \max\left\{\lambda_f[J(a) - V(a)] + V'(a), 0\right\}, & a \le \bar{a}; \\ \max\left\{\lambda_f[J(0) - I_u^J(a) - V(a)] + V'(a), 0\right\}, & \bar{a} < a \le \hat{a}; \end{cases} \tag{11}$$

$$(r-g)U = b + \int_0^{\bar{a}} \lambda_w(a)[W(a) - U]da + \int_{\bar{a}}^{\hat{a}} \lambda_w(a)\big[W(0) - I_u^W(a) - U\big]da. \tag{12}$$

Vacant machines that are older than the critical age $\bar{a}$ do not exit, but wait until they meet a worker and then upgrade. At the time the machine is upgraded, the firm pays its share of the upgrading cost $I_u^J(a)$. When age $\hat{a}$ is reached, upgrading becomes infinitely costly, so the vacant firm exits with value equal to zero. The return for unemployed workers contains an additional term that gives the value of meeting vacant machines older than $\bar{a}$. These firms will upgrade upon meeting a worker, and the worker contributes $I_u^W(a)$ to the upgrading cost and starts working with brand-new capital.

Wage Determination. In the presence of frictions, a bilateral monopoly problem between the firm and the worker arises and, thus, wages are not competitive. As is standard in the literature, we choose a cooperative Nash-bargaining solution for wages. In particular, we assume that the parameter defining the relative bargaining power is the same when the pair negotiates over how to split output and over how to share the upgrading cost I_u. With outside options as in the previous equations, the wage is such that at every instant a fraction β of the total surplus

13. This also means that a matched machine of age $\bar{a}$ or higher would be scrapped, since there is no cost for a vacant machine of waiting for a worker. Here the absence of a vacancy-posting cost simplifies the nature of the equilibrium. With such a cost, unmatched firms would scrap capital at an earlier age than would matched firms.

$S(a) \equiv J(a) + W(a) - V(a) - U$ of a type a match goes to the worker and a fraction $(1 - \beta)$ goes to the firm:

$$W(a) = U + \beta S(a) \quad \text{and} \quad J(a) = V(a) + (1 - \beta)S(a). \tag{13}$$

Using the surplus-based definition (13) of the value of an employed worker $W(a)$ in equation (8) and rearranging terms, we obtain the wage rate as

$$w(a) = (r - g)U + \beta[(r - g + \sigma)S(a) - S'(a)]. \tag{14}$$

Allocation of Upgrading Costs. The allocation of gains from upgrading is assumed to maximize the joint surplus.[14] Firms of age a and workers thus split the gain from upgrading

$$G(a) \equiv J(0) + W(0) - I_u - J(a) - W(a), \tag{15}$$

according to the surplus sharing rule with parameter β. Thus, they jointly solve

$$\begin{aligned} &\max_{I_u^W(a), I_u^J(a)} [J(0) - J(a) - I_u^J(a)]^\beta \big[W(0) - W(a) - I_u^W(a)\big]^{1-\beta} \\ &\text{subj. to:} \qquad I_u^J(a) + I_u^W(a) = I_u. \end{aligned} \tag{16}$$

The upgrading cost is then distributed according to

$$\begin{aligned} I_u^W(a) &= W(0) - W(a) - \beta G(a), \\ I_u^J(a) &= J(0) - J(a) - (1 - \beta)G(a). \end{aligned} \tag{17}$$

4. Characterizing the Stationary Equilibrium of the Frictional Economy

We characterize the equilibrium of the matching model in terms of two variables: the age at which a firm exits the market or upgrades its machine and the rate at which vacant firms meet workers: $(\bar{a}, \lambda_f)$. The two variables are jointly determined by two key conditions. The first condition is labeled the job destruction or job upgrading condition. In the economy with creative destruction (upgrading) this condition expresses the indifference between carrying on and scrapping (upgrading) the machine for a match with capital of age $\bar{a}$. The second condition, labeled the job creation condition, expresses the indifference for outside firms between creating a vacancy with the newest vintage and not entering. This characterization is conditional on the steady-state employment and vacancy distributions. In Section 4.3 we show how these distributions can be characterized in terms of the two unknowns $(\bar{a}, \lambda_f)$.

14. In an earlier version of the paper, we also considered another assumption, namely that the firm bears all of the upgrading costs and maximizes its own gains from upgrading. The quantitative difference in results was small.

4.1. The Economy with Creative Destruction

The Surplus Function. It is useful to start by stating the (flow version of the) surplus equation. Using the definition of the surplus $S(a)$ we arrive at

$$(r-g)S(a) = \max\{e^{-\varphi a} - \sigma S(a) - \lambda_f(1-\beta)S(a) - (r-g)U + S'(a), 0\}. \tag{18}$$

This asset-pricing-like equation is obtained by combining equations (7), (8), (9), and (13): the growth-adjusted return on surplus on the left-hand side equals the flow gain on the right-hand side, where the flow gain is the maximum of zero and the difference between total inside minus total outside flow values. The inside value includes: a production flow $e^{-\varphi a}$, a flow loss due to the probability of a separation of the match $\sigma S(a)$, and changes in the value for the matched parties, $J'(a) + W'(a)$. The outside option flows are: the flow gain from the chance that a vacant firm matches $\lambda_f(1-\beta)S(a)$, the change in the value for the vacant firm $V'(a)$, and the flow value of unemployment $(r-g)U$. Note a key difference with the traditional model: the value of a vacancy is positive, and it contributes towards a reduction of the rents created by the match.[15]

The solution of the first-order linear differential equation (18) is the function

$$S(a) = \int_a^{\overline{a}} e^{-(r-g+\sigma+(1-\beta)\lambda_f)(\tilde{a}-a)}\left[e^{-\phi\tilde{a}} - (r-g)U\right]d\tilde{a}. \tag{19}$$

We have used the boundary condition associated with the fact that the surplus-maximizing decision is to keep the match alive until an age $\bar{a}$ when there is no longer any surplus to the match, $S(\bar{a}) = 0$, and there is no gain from a marginal delay of the separation, $S'(\bar{a}) = 0$. For lower a's, the match will have strictly positive surplus, and for values of a above $\bar{a}$, the surplus will be equal to zero.[16] Intuitively, the surplus is decreasing in age a for two reasons: first, the time-horizon over which the flow surplus accrues to the pair shortens with a; and second, the value of a job's output declines with age relative to that of the new vacant jobs.

Equation (19) contains a non standard term associated with the non degenerate distribution of vacancies: the non-zero firm's outside option of remaining vacant with its machine reduces the surplus by increasing the "effective" discount rate through the term $(1-\beta)\lambda_f$. Everything else being equal, the quasi-rents in the match are decreasing as the bargaining power of the idle firm or its meeting rate is increasing.

15. In equation (18) we have $S'(a) = J'(a) + W'(a) - V'(a)$.

16. Straightforward integration of the right-hand side in (19) and further differentiation shows that, over the range $[0, \bar{a})$, the function $S(a)$ is strictly decreasing and convex; moreover, $S(a)$ will approach 0 for $a \to \bar{a}$.

The Job-Destruction Condition. The optimal separation rule $S'(\bar{a}) = 0$ together with equation (19) implies that the exit age $\bar{a}$ satisfies

$$e^{-\varphi \bar{a}} = (r - g)U, \tag{20}$$

for a given value of unemployment U. The left-hand side of (20) is the output of the oldest match in operation, whereas the right-hand side is the flow-value of an idle worker. The idea is simple: firms with old enough capital shut down because workers have become too expensive, since the average productivity of vacancies and, therefore, the workers' outside option of searching, is growing at a constant rate. Note that this equation resembles the profit-maximization condition in the frictionless economy, with the worker's flow outside option, $(r - g)U$, playing the role of the competitive wage rate. In fact, from the wage equation (14), it follows that the lowest wage paid in the economy (on machines of age $\overline{a}$) exactly equals the flow value of unemployment.

We can now rewrite the surplus function (19) in terms of the two endogenous variables $(\bar{a}, \lambda_f)$ only, by substituting for $(r - g)U$ from (20):

$$S(a; \bar{a}, \lambda_f) = \int_a^{\bar{a}} e^{-(r-g+\sigma+(1-\beta)\lambda_f)(\tilde{a}-a)} \big[e^{-\varphi \tilde{a}} - e^{-\varphi \bar{a}}\big] d\tilde{a}. \tag{21}$$

In this equation, and occasionally below, we use a notation of values (the surplus in this case) that shows an explicit dependence of $\bar{a}$ and λ_f. From (21) it is immediately clear that $S(a; \bar{a}, \lambda_f)$ is strictly increasing in $\bar{a}$ and decreasing in λ_f. A longer life span of capital $\bar{a}$ increases the surplus at each age because it lowers the flow value of the worker's outside option, as evident from (20). A higher rate at which firms, when idle, meet workers reduces the surplus because it increases the outside option for a firm and shrinks the rents accruing to the matched pair.

Using (10), (13), and (20) we obtain the optimal separation (or *job-destruction*) condition

$$e^{-\varphi \bar{a}} = b + \beta \int_0^{\bar{a}} \lambda_w(a; \bar{a}, \lambda_f) S(a; \bar{a}, \lambda_f) da, \tag{JD}$$

which is an equation in the two unknowns $(\bar{a}, \lambda_f)$. The rates $\lambda_w(a)$ at which unemployed workers are matched with firms also depend on the two endogenous variables; the explicit dependence on $\bar{a}$ and λ_f is described below.

The Job-Creation Condition. We define the value of a vacancy of age a using the new expression (21) for the surplus of a match $S(a; \bar{a}, \lambda_f)$ together with (13). The differential equation for a vacant firm (9) then implies that the net-present-value of a vacant firm equals

$$V(a; \bar{a}, \lambda_f) = \lambda_f(1-\beta)\int_a^{\bar{a}} e^{-(r-g)(\tilde{a}-a)} S(\tilde{a}; \bar{a}, \lambda_f) d\tilde{a}, \tag{22}$$

where $\bar{a}$ equals the age at which the vacant firm exits. Since vacant firms do not incur any direct search cost, they will exit the market at an age such that this expression equals zero. This immediately implies that vacant firms will exit at the same age $\bar{a}$ at which matched firms exit and scrap their capital. Since in equilibrium there are no profits from entry, we must have that $V(0; \bar{a}, \lambda_f) = I_0$, and we thus have the free-entry (or *job creation*) condition

$$I_0 = \lambda_f(1-\beta)\int_0^{\bar{a}} e^{-(r-g)a} S(a; \bar{a}, \lambda_f) da. \tag{JCD}$$

This condition requires that the cost of creating a new job I_0 equals the value of a vacant firm at age zero, which is the expected present value of the profits it will generate: a share $1-\beta$ of the discounted future surpluses produced by a match occurring at the instantaneous rate λ_f. The job creation condition is the second equation in the two unknowns $(\bar{a}, \lambda_f)$.

In Hornstein, Krusell, and Violante (2003), we demonstrate that a solution to the two equations (JCD) and (JD) in the pair $(\bar{a}, \lambda_f)$ exists and is unique under general conditions: in particular, the (JCD) condition traces a strictly decreasing curve in the $(\bar{a}, \lambda_f)$ space, whereas the (JD) condition traces a strictly increasing relationship.

4.2. The Economy with Upgrading

We now consider an economy where the firm and worker jointly decide on when the machine should be upgraded and both parties share in the cost of the project.

Optimal Upgrading. A worker-firm pair will not upgrade its machine as long as the gains from upgrading are negative. The match values, being the discounted expected present values of future returns, are continuous functions of the age. Since upgrading is instantaneous, at the time a machine is upgraded the gain from upgrading is then zero:

$$G(\bar{a}) = J(0) + W(0) - I_u - J(\bar{a}) - W(\bar{a}) = 0. \tag{23}$$

At the optimal upgrading age not only is the gain from upgrading zero, but so is the marginal gain from a delay of the upgrading decision. This means that at the upgrading age the derivative of the gain function is zero, that is, $J'(\bar{a}) + W'(\bar{a}) = 0$. We can use this condition when we add the value function definitions of matched workers, (8), and firms, (7):

$$(r-g)[J(\bar{a}) + W(\bar{a})] \\ = e^{-\varphi\bar{a}} - \sigma[J(\bar{a}) + W(\bar{a}) - V(\bar{a}) - U] + J'(\bar{a}) + W'(\bar{a}). \quad (24)$$

Together, the two no-gains conditions then imply

$$(r-g)[J(0) + W(0) - I_u] = e^{-\phi\bar{a}} - \sigma[J(0) + W(0) - I_u - V(\bar{a}) - U]. \quad (25)$$

This condition states that at the optimal upgrading age $\bar{a}$ the firm-worker pair is indifferent between an upgrade and a marginal delay of the upgrade. The left-hand side of this expression is the matched pair's return on an upgraded machine at $\bar{a}$, and the right-hand side is the flow return from a marginal delay of the upgrading decision: the production flow minus the surplus capital loss from delay due to separation.[17]

Optimal upgrading depends on the value of a vacancy at the upgrading age, which in turn depends on the expected present value of a vacant firm's gain from upgrading upon meeting a worker (11). From the rule (17), which determines how the gains from upgrading (15) are shared, we obtain

$$J(0) - I_u^J(a) - V(a) = (1-\beta)[S(0) + I_0 - I_u - V(a)], \quad (26)$$

where we have used the free-entry condition $V(0) = I_0$.[18] Substituting this expression in the definition of the value of a vacancy (11) for $\bar{a} \leq a \leq \hat{a}$, collecting terms, and solving the differential equation subject to the terminal condition $V(\hat{a}) = 0$, we obtain an expression for the vacancy value for $a \geq \bar{a}$

$$V(a) = \kappa_V(a, \lambda_f)[S(0) + I_0 - I_u], \quad (27)$$

with $\kappa_V(a, \lambda_f) = [1 - e^{-\rho_V(\hat{a}-a)}](1-\beta)\lambda_f/\rho_V$ and $\rho_V = (1-\beta)\lambda_f + r - g$. In an equilibrium the value of a vacancy is non negative at the upgrading age.

The surplus and the vacancy value at the upgrading age and the surplus and vacancy value for a firm with a new machine differ only through the cost of upgrading. To see this, use the surplus-sharing rule (13) and the free-entry condition in the no-gains condition (23) and we arrive at

$$S(\bar{a}) + V(\bar{a}) = S(0) + I_0 - I_u. \quad (28)$$

17. In Appendix B we provide a heuristic derivation of this indifference condition based on the limit of discrete time approximations.

18. Because we have imposed a maximum age for capital, there will always be machine exit, so entry must occur in any steady state.

This means that the surplus at the upgrading age is given by

$$S(\bar{a}) = [1 - \kappa_V(\bar{a}, \lambda_f)][S(0) + I_0 - I_u]. \tag{29}$$

In an equilibrium, both the surplus and vacancy value at the upgrading age are nonnegative. Note that in the economy with upgrading, the surplus and vacancy value at the upgrading age can be strictly positive, unlike in the creative destruction economy where machines are scrapped at the exit age because the surplus of the match is zero. Since $\kappa_V \in (0, 1)$ either the surplus and vacancy value are both zero or both strictly positive.

In the definition of the job-upgrading and job-creation conditions next, we use the surplus function $S(a; \bar{a}, \lambda_f)$, defined for machines that have not yet reached the upgrading age: $0 \leq a \leq \bar{a}$. In Appendix C we show that we can write the surplus as a function of the upgrading age and the worker meeting rate only. As a first step towards that result, we derive an expression for the surplus value of a new machine as a function of $(\bar{a}, \lambda_f)$ only, $S(0; \bar{a}, \lambda_f)$.

The Job-Creation Condition. The condition that ensures zero profits at entry is always $I_0 = V(0; \bar{a}, \lambda_f)$ but we now have a different expression for the value of a vacant job. From (11) and (13), it is easy to see that, for $a \leq \bar{a}$,

$$\begin{aligned} V(a; \bar{a}, \lambda_f) = \lambda_f(1 - \beta) \int_a^{\bar{a}} e^{-(r-g)(\tilde{a}-a)} S(\tilde{a}; \bar{a}, \lambda_f) d\tilde{a} \\ + e^{-(r-g)(\bar{a}-a)} V(\bar{a}; \bar{a}, \lambda_f). \end{aligned} \tag{30}$$

Since the cost of upgrading a machine is independent of the age as long as the machine is not too old, $a \leq \hat{a}$, vacant machines older than $\bar{a}$ will be upgraded. Vacant machines at the upgrading age $\bar{a}$ therefore tend to have positive value, unlike vacant machines at the exit age in a creative-destruction economy. We can substitute (27) for the vacancy value at $\bar{a}$ and obtain the job-creation equilibrium condition as a function of the pair of unknowns $(\bar{a}, \lambda_f)$ only:

$$\begin{aligned} I_0 = \lambda_f(1 - \beta) \int_0^{\bar{a}} e^{-(r-g)a} S(a; \bar{a}, \lambda_f)\, da \\ + e^{-(r-g)\bar{a}} \kappa_V(\bar{a}, \lambda_f)[S(0; \bar{a}, \lambda_f) + I_0 - I_u]. \end{aligned} \tag{JCU}$$

Also this equation is easily comparable with the job-creation condition (JCD) of the creative-destruction model, since it features only one additional positive term.[19] This term captures the additional value of using the upgrading option

19. Note however that the expression for the surplus function $S(a; \bar{a}, \lambda_f)$ and the distributions $\lambda_w(a; \bar{a}, \lambda_f)$ are not the same in the two economies. We discuss the invariant distributions in Section 4.3, and as noted before, we derive the expression for the surplus in Appendix C.

when opening a job. This value is decreasing in I_u and also decreasing in λ_f; the reason is that the marginal gain of the upgrading option, compared to scrapping, is large if the firm's meeting process is slow.

The Job-Upgrading Condition. From the condition for the optimal delay of upgrading (25), together with the expression for the vacancy value at the threshold age $\bar{a}$ (27) and the surplus definitions (13), it is easy to derive that

$$e^{-\varphi\bar{a}} - \kappa_J(\bar{a}, \lambda_f)[S(0; \bar{a}, \lambda_f) + I_0 - I_u] = (r - g)U, \tag{31}$$

with $\kappa_J(\bar{a}, \lambda_f) = r - g + \sigma[1 - \kappa_V(\bar{a}, \lambda_f)]$. Now consider the flow value of unemployment (12). Using the surplus-sharing rules (13) in the first integral term and the upgrading cost-sharing rule (17) in the second integral term of the right-hand side, and substituting expression (31) for the flow return on unemployment on the left-hand side, we obtain an expression for the job upgrading condition entirely as a function of $(\bar{a}, \lambda_f)$:

$$\begin{aligned} e^{-\varphi\bar{a}} &= b + \beta \int_0^{\bar{a}} \lambda_w(a; \bar{a}, \lambda_f) S(a; \bar{a}, \lambda_f)\, da + \big[S(0; \bar{a}, \lambda_f) + I_0 - I_u\big] \\ &\quad \times \left\{ \kappa_J(\bar{a}, \lambda_f) + \beta \int_{\bar{a}}^{\hat{a}} \lambda_w(a; \bar{a}, \lambda_f)[1 - \kappa_V(a, \lambda_f)]\, da \right\}. \end{aligned} \tag{JU}$$

Comparing this equation with the job destruction condition in the economy with creative destruction (JD), we note again an additional term, always positive, implying that the upgrading age $\bar{a}$ is lower than the destruction age; how much lower it is depends on the size of the extra term.

4.3. Invariant Employment and Vacancy Distributions

The two kinds of economies we are considering—one with creative destruction and one with upgrading—have many parts in common but the distributions of matched firm/worker pairs are quite different. In particular, in the former economy the density of matched pairs is sharply increasing in the age of the capital, unlike in the economy with upgrading. This is because new capital must be created prior to the creation of the match. We now describe how to construct the employment and vacancy distributions in some more detail. This characterization is interesting for understanding wage inequality, but it is also important for finding explicit expressions for the matching probabilities in terms of the endogenous variables $(\bar{a}, \lambda_f)$.

The Economy with Creative Destruction. Denote with $\mu(a)$ the measure of matches between an a firm and a worker, and denote total employment with μ.[20] The inflow of new firms is $v(0)$: new firms acquire the new capital and proceed to the vacancy pool. Thereafter, these firms transit stochastically back and forth between vacancy and match: firms are matched with workers at rate λ_f and they become vacant at rate σ. Finally firms exit at age $\bar{a}$, whether vacant or matched. The evolution of employment and vacancies in a stationary distribution is then determined by the differential equations

$$v'(a) = \sigma\mu(a) - \lambda_f v(a), \text{ for } 0 \leq a \leq \bar{a} \tag{32}$$

$$\mu'(a) = \lambda_f v(a) - \sigma\mu(a), \text{ for } 0 \leq a \leq \bar{a}. \tag{33}$$

The evolution of matched machines is the mirror image of the evolution of vacancies, i.e., $\mu'(a) = -v'(a)$. This implies that the number of vacant and matched machines of age a less than $\bar{a}$ remains constant:

$$v(a) + \mu(a) = v(0) + \mu(0), \text{ for } 0 \leq a \leq \bar{a}. \tag{34}$$

For $a \in [0, \bar{a})$, the evolution of $\mu(a)$ therefore follows

$$\mu'(a) = \lambda_f[v(0) + \mu(0)] - (\sigma + \lambda_f)\mu(a). \tag{35}$$

Because all firms proceed first to the search pool with their new machines, we can solve this differential equation subject to the initial condition $\mu(0) = 0$. As Figure 1 reveals, the employment (vacancy) density is increasing and concave (decreasing and convex) in age a. The reason for this is that for every age $a \in [0, \bar{a})$ there is a constant number of machines, and older machines have a larger cumulative probability of having been matched in the past. This feature distinguishes our model from standard-search vintage models where the distribution of vacant jobs is degenerate at zero and the employment density is decreasing in age a at a rate equal to the exogenous destruction rate σ.

With the vacancy distribution in hand, we now have the explicit expression for the value of $\lambda_w(a)$,

$$\lambda_w(a; \bar{a}, \lambda_f) = \lambda_w \frac{\nu(a)}{v} = m(\theta, 1)\frac{\sigma + \lambda_f e^{-(\sigma+\lambda_f)a}}{\bar{a}\sigma + \frac{\lambda_f}{\sigma+\lambda_f}\left(1 - e^{-(\sigma+\lambda_f)\bar{a}}\right)}, \tag{36}$$

which depends only on the pair of endogenous variables $(\bar{a}, \lambda_f)$, given the strictly decreasing relation between θ and λ_f, equation (5).[21]

20. In Appendix D we derive the differential equations that characterize the stationary employment dynamics, equations (32) and (33).

21. The closed-form expressions for employment and vacancy densities, (E.2) and (E.3), are in Appendix E.

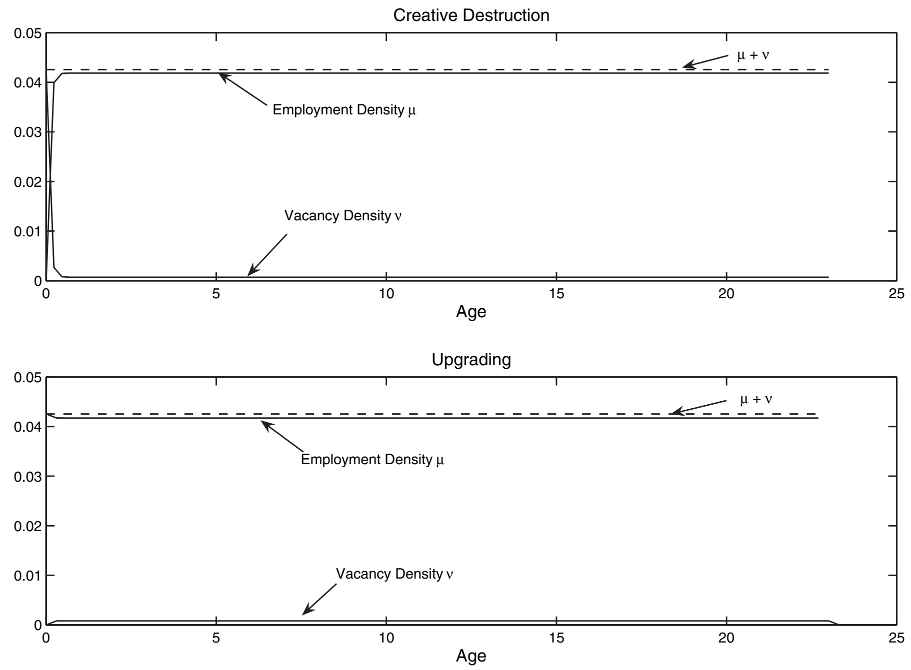

FIGURE 1. Employment and vacancy densities.

The Economy with Upgrading. In the economy with upgrading, the key difference is that after the machine of a firm reaches age $\bar{a}$, the firm does not exit, but it upgrades the machine if it is matched to a worker and thereby resets the age to 0. Only vacant firms that do not meet a worker by age $\hat{a}$ exit the economy. For firms with relatively young machines, i.e., firms that do not upgrade their machines, the evolution of employment and vacancies in a stationary distribution continues to be determined by the differential equations (32) and (33), and equation (34). With the possibility of the instantaneous upgrading of machines at age $\bar{a}$ and ongoing upgrading of vacant machines with age $a \geq \bar{a}$, the employment density of new machines is now strictly positive:

$$\mu(0) = \mu(\bar{a}) + \lambda_f \int_{\bar{a}}^{\hat{a}} v(a)da. \tag{37}$$

This means that in the economy with upgrading, the employment density is maximal for the newest vintage—it contains the machines which have just upgraded in addition to all vintages which immediately upgrade upon meeting a worker. Figure 1 illustrates how the two economies differ. Finally, the evolution of vacancies that are older than the upgrading age is given by

$$v'(a) = -\lambda_f v(a), \text{ for } \bar{a} < a \leq \hat{a}. \tag{38}$$

Since matched machines never reach age $a > \bar{a}$ there are no additions from exogenous separations. Once the machine of a vacant firm reaches age $\hat{a}$, upgrading becomes infeasible and the firm exits.

In Appendix E we show how to solve the system of equations (32), (33), (34), (37), and (38) for the employment and vacancy distribution, $\mu(a)$ and $v(a)$, conditional on the pair $(\bar{a}, \lambda_f)$. With the vacancy distribution in hand, we use (6) to obtain $\lambda_w(a)$, i.e., the rate at which workers meet vacant firms.

5. The Quantitative Analysis

The calibration proceeds as follows. For both the creative-destruction economy and for the upgrading economy, we target a common set of steady-state aggregate variables not including any measure of wage inequality. We then look at the implications for the level of frictional wage inequality given the resulting parameterizations. Next, we look at the effect of changing the rates of disembodied and embodied technology growth on unemployment and other equilibrium labor market outcomes.

5.1. Calibration

For each of the two model economies we choose the parameter values to match the same set of steady-state values based on the U.S. economy. We choose a Cobb–Douglas specification for our matching function, $m = \kappa v^{\alpha} u^{1-\alpha}$, where κ is the scale parameter. Hence, overall, we have 13 parameters to calibrate: $(r, \kappa, \alpha, \sigma, \beta, \delta, I_0, I_u, \hat{a}, \psi, \omega, \gamma, b)$.

We set r to match an annual interest rate of 4%. We set the matching elasticity with respect to vacancies, α, to 0.5, an average of the values reported in the comprehensive survey of empirical estimates of matching functions by Petrongolo and Pissarides (2001, Table 3).

We have two sources of growth in the model: disembodied productivity change, occurring at rate ψ, and capital-embodied productivity change. Hornstein and Krusell (1996) measure annual disembodied growth in the United States for 1954–1993 to be 0.8% per year, whereas more recently, Cummins and Violante (2002) compute it to be 0.3% per year from 1965–2000. We set $\psi = 0.5\%$. At least since Greenwood, Hercowitz, and Krusell (1997), a number of authors have suggested to measure the speed of embodied technical change through the (inverse of the) rate of decline of the quality-adjusted relative price of capital. We argued before that in our environment, embodied technical change is directly reflected in the rate at which the relative price of new capital declines, $\gamma - g$. Gordon's (1990) influential work on quality-adjusted prices for durable goods suggests a value for the annual rate of embodied technical change in the United States around 3%. Given the observed 2% average U.S. output growth rate g, the rate of price decline for capital implies that $\gamma = 5\%$. From the relation $g - \gamma = \psi - (1 - \omega)\gamma$, we obtain a capital elasticity parameter $\omega = 0.3$.

For the creative-destruction economy we simultaneously calibrate the remaining parameters $(\sigma, \beta, \delta, I_0, \kappa, b)$ so that the steady state has (1) an unemployment rate of 4% (the U.S. historical average); (2) an average unemployment duration of 8–9 weeks, as reported by Abraham and Shimer (2001); (3) a labor income share of 0.685 (Cooley and Prescott 1995); (4) an average vacancy duration of four weeks as estimated by Hall (2003) based on data from the Job Openings and Labor Turnover Survey; (5) an average age of capital of about 11.5 years, as reported by the Bureau of Economic Analysis (2002); and (6) an average replacement rate of 10%.[22]

The parameter b is supposed to summarize a wide range of benefit policies that vary with unemployment duration and family status (none of which we model). The OECD Employment Outlook (1996) provides replacement rates for unemployment benefits in OECD countries from 1961 to 1995 for two earnings levels, three family types, and three durations of unemployment. The reported average replacement rate (in terms of the average wage) for the United States in that period was 10% and we choose b to replicate this number.

For the economy with upgrading one can think of the calibration as matching the same targets but using I_u instead of I_0. This is because with upgrading we set the maximal age $\hat{a}$ beyond which machines cannot be upgraded at 30 years. Given the matched steady-state rate at which vacant machines meet workers, this high value of $\hat{a}$ means that almost all machines will eventually upgrade even if initially vacant, and entry of new machines will essentially be zero. Since effectively there is no entry, $\hat{a}$ and I_0 only have a minor effect on the steady state (so I_0 can be solved for residually): we again match six variables with six parameters.

The parameter values are summarized in Table 1. A clear conclusion emerges: if we take the distance between parameters as a measure of closeness of the two economies, then it appears that the economies are remarkably similar: very small parametric differences are needed to match the same set of facts.

In our calibration we have not tried to match the wage inequality generated by the vintage induced productivity differences in the two economies. We use two measures of wage inequality: the 90–10 wage ratio used widely in the empirical labor literature, and the ratio of average wages to minimum wages.[23] From

22. The unemployment rate together with the average unemployment duration imply an annual separation rate for workers from employment to unemployment of 22% for either economy. This value of the separation rate is in line with the data reported in Alogoskoufis et al. (1995, p. 10).

23. The latter measure is useful because we show in Hornstein, Krusell, and Violante (2005) that the standard DMP model without capital makes very tight predictions on what this measure must be given the usual steady-state statistics.

TABLE 1. Calibration of benchmark economies.

	Economy with upgrading investment	Economy without upgrading investment
Common parameters $r = 0.04,\ \alpha = 0.5,\ \psi = 0.005,\ \gamma = 0.05,\ \omega = 0.3$		
Model-specific parameters		
κ	8.911	8.923
δ	0.061	0.064
σ	0.255	0.214
β	0.802	0.767
b	0.048	0.048
I_0	4.599	4.281
I_u	4.278	NA
$\hat{a}$	30.000	NA
Wage inequality		
w_{90}/w_{10}	1.078	1.058
$w_{average}/w_{\min}$	1.053	1.040

Table 1 we can see that the 90–10 wage ratio is about 8% (6%) in the economy with(out) upgrading, and the second inequality measure is about 5% (6%) in the economy with(out) upgrading. The fact that the economy with upgrading generates somewhat more wage inequality for both measures is related to the qualitative difference of employment densities discussed in Section 4.3. Even though the maximal and minimal wages of the youngest and oldest vintages are very similar in the two economies, there are relatively more high-productivity young vintages in the economy with upgrading. This feature increases the upper cut-off wage for the 90–10 wage ratio and it increases average wages in the economy with upgrading.

To put these numbers on wage inequality in perspective, for observationally equivalent workers Katz and Autor (1999) report 90–10 wage ratios around 1.5 and Hornstein, Krusell, and Violante (2005) report ratios of mean wages to minimal wages (proxied by the first percentile) around 1.3. These numbers are computed on the distribution of residuals in a typical wage regression, after controlling for observable characteristics of the workers (gender, race, education, experience) and fixed individual effects to capture "unobserved ability." Hence, the combination of matching frictions ("luck") and vintage capital differentials can only explain at best one-sixth of the observed inequality among ex ante equal workers.

5.2. Creative-Destruction versus Upgrading: A Comparison

We now analyze, for the baseline parametrization, the response of the two economies to accelerations in the rate of embodied technical change, γ, and in the rate of disembodied technical change, ψ, of an empirically plausible magnitude. In our analysis we focus on the behavior of the unemployment rate, the average unemployment duration, the critical age at which upgrading/exit occurs, the wage income share, the ratio of average wages to lowest wages, and the 90–10 wage ratio.

Embodied Technical Change. Krusell et al. (2000) and, more recently, Cummins and Violante (2002) have argued that the annual rate of embodied technical change in the United States has increased substantially in the past two decades, up to 6.5% over the years 1995–2000.[24] This estimate, together with the assumption that ψ and ω remain constant, means that γ has increased to 10% so as to generate a decline in the relative price of capital of 6.5% per year.

The results of this experiment are reported in Figure 2. A faster rate of embodied technical change increases the unemployment rate and wage inequality and lowers the wage income share. The accelerated technical change shortens the

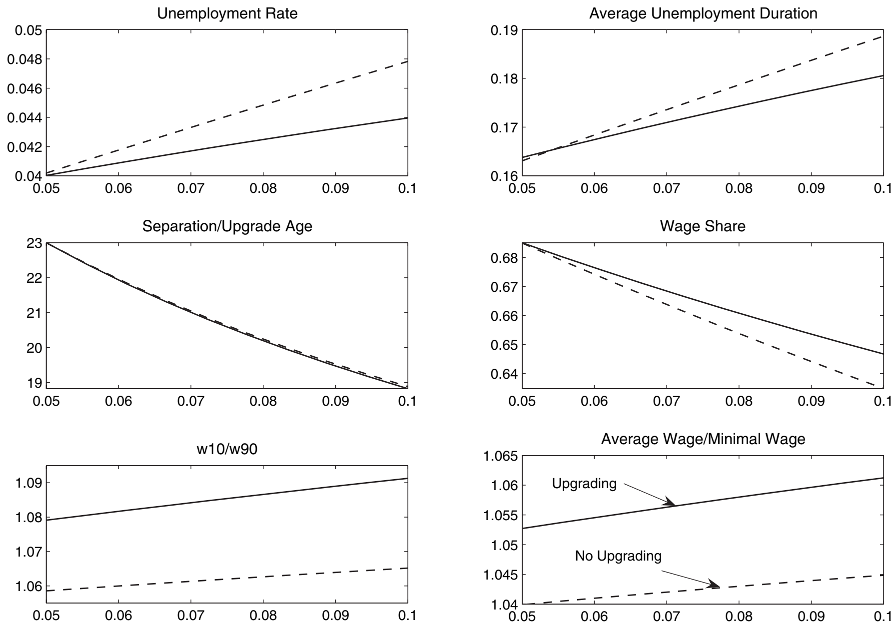

FIGURE 2. Comparative statics for embodied technical change γ.

24. Other authors, using measurement techniques different from quality-adjusted relative prices, arrived at very similar conclusions on the pace of embodied technical change in the postwar era (see for example, Hobijn 2000) for the United States.

useful lifetime of a machine, that is, machines are either upgraded at a faster rate or they exit the economy at a faster rate. Although wages fall, reflected in the declining wage income share, they do not fall enough to compensate completely for the shortened lifetime of machines. As a consequence, the value of firms declines, but in equilibrium the value of a new machine has to equal the constant (normalized) cost of a new machines. Therefore, the rate at which firms meet workers has to increase, and correspondingly the rate at which workers meet firms declines, that is the average duration of unemployment increases. This means that the measure of active firms and employment declines.

Employment in the economy without upgrading declines somewhat more than in the economy with upgrading because of the creative destruction effect: upon firm exit workers enter the unemployment pool. Overall, the differences across the two economies are minor along all dimensions examined.

Disembodied Technical Change. Mortensen and Pissarides (1998) have pointed out that there is a qualitative difference between embodied and disembodied technical change. Whereas a higher rate of embodied technical change tends to lower the value of existing machines, a higher rate of disembodied technical change increases the value of machines because it increases the output over the lifetime of a machine. Machines become more valuable, vacancy values increase, more machines seek to match with workers, and therefore unemployment declines. Wage inequality declines and the wage income share increases.

From Figure 3 we see (1) that the economies with and without upgrading essentially respond in the same way to a change in the rate of disembodied technical change, and (2) that doubling the rate of disembodied technical change has a negligible quantitative effect on labor market variables.

5.3. Interpreting the Findings: An Intuitive Argument

The facts that search frictions generate only a limited amount of wage inequality in our vintage model and that the nature of the investment decision does not affect the comparative statics in a quantitatively important way, result from two important quantitative restrictions that calibration to the U.S. labor market imposes on the equilibrium of our model.

First, on average a job/machine remains vacant in the U.S. economy for about four weeks, to be compared to the expected lifetime of a new machine, which is over 11 years. When the labor market frictions are so small from the firm's perspective, the firm becomes roughly indifferent between scrapping and upgrading: for a given cost of new machines, the upgrading option is substantially better than scrapping only if the matching process is long and costly.

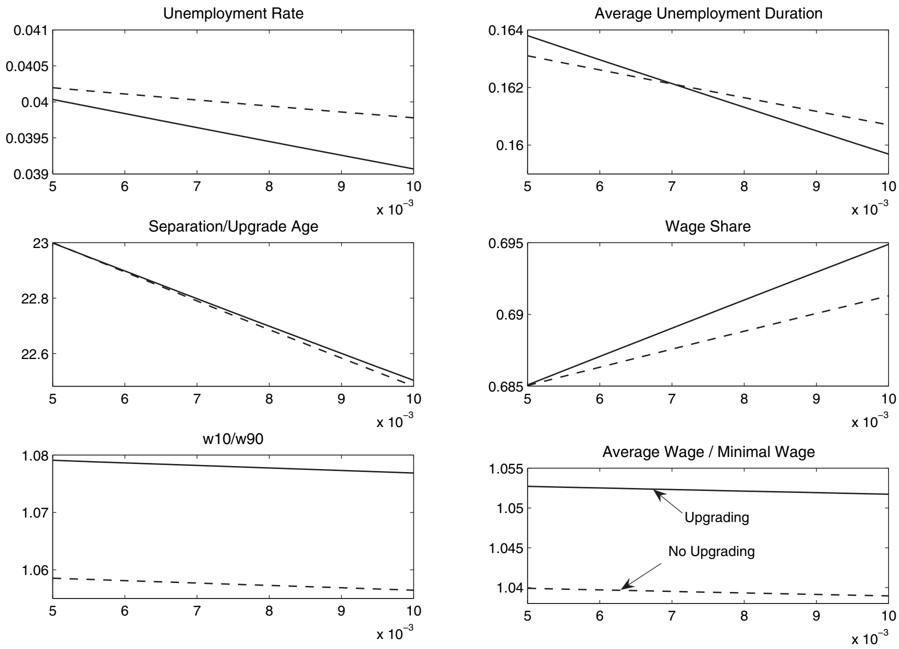

FIGURE 3. Comparative statics for disembodied technical change ψ.

Second, the average duration of unemployment in the U.S. economy is very short, around eight weeks. In search models, unemployment duration is directly related to the option value of search for the risk-neutral worker and the latter depends positively on wage dispersion. A duration of unemployment as short as it is in the data can therefore only be consistent with a minor amount of frictional inequality.

Interestingly, our results are consistent with the conclusions of Ridder and van den Berg (2003) who estimate the severity of search frictions in the United States, the United Kingdom, France, Germany, and the Netherlands and find that the labor market in the United States is, by far, the least frictional. The (inverse of their) measure of search frictions—the arrival rate of job offers to the unemployed workers—for the United States is 5 times larger than for the United Kingdom, 8 times larger than for Netherlands, 15 times larger than for France, and 22 times larger than for Germany.

Technically, we show in Proposition (1) below that, for $I_0 = I_u$ and as matching frictions vanish, the upgrading model collapses to the creative-destruction economy. In addition, both economies converge to the frictionless benchmark of Section 2 where the equilibrium displays a unique market clearing wage.

PROPOSITION 1. *If $I_0 = I_u$ and the meeting rate in the labor market becomes infinitely high ($\lambda_f \to \infty$), the economy with upgrading converges to the economy with creative destruction. Moreover, the two economies converge to the frictionless benchmark which displays no wage inequality.*

Proof. See Appendix F.

Finally, note that the result that the frictionless vintage capital model is obtained as the limit when frictions vanish is a desirable and natural property of our setup, which is not shared by the traditional framework. It is easy to verify that the equilibrium of the traditional matching model with vintage capital, where vacant firms do not purchase a machine upon entry but simply pay a flow cost to post their vacancy (recall our discussion of Section 3), converges in the limit to an economy where only the newest type of capital is in operation. Put differently, not only do wage inequality and unemployment disappear as the matching frictions fade, which is natural, but the vintage capital component of the model vanishes as well.

Therefore, in the traditional model the vintage capital structure is not a result of the nature of technical change and of the investment decision, but its existence depends entirely on the fact that it is costly for firms to replace workers, and hence that firms hold on to their machines longer. This latter effect is still present in our model, but it is not the only force generating heterogeneity in the age of capital in operation.

6. Extensions

Still within the context of identical workers, we now consider two additional sources of inequality: productivity heterogeneity that is (i) *machine-specific* or is (ii) *match-specific*. In the former case, an entrepreneur who enters the economy buys a new machine but the new machine may not be a good match with the entrepreneur's skills, and independently of the worker with whom the entrepreneur-machine pair is matched, the idiosyncratic productivity level stays intact (controlling for the aging of capital). Thus, the relevant idiosyncratic specificity is between the entrepreneur and the machine; once the machine is scrapped (or upgraded), the specificity disappears. In the latter case, in contrast, the relevant idiosyncratic specificity is between the entrepreneur-machine and the worker; once the worker is gone, the specificity disappears.

In the economy with match-specific productivity, upgrading (as opposed to creative destruction) helps firm-worker pairs keep the good matches for a longer time. In the economy with machine-specific productivity, this effect is not present.

6.1. Model Specifics

We now briefly outline some specifics of each of the extensions. The details of each model can be found in the Appendix.

Machine-Specific Productivity. In the economy with machine-specific productivity, once a new machine is created it has initial productivity e^z, which is drawn from a probability density function $f(z)$ with bounded support $[0, \hat{z}]$. As before, machines age and depreciate such that the productivity of an age a machine at time t relative to a new machine with initial productivity e^{z+gt} is $e^{z-\varphi a}$. This productivity does not depend on the worker the machine/firm is matched with. If the machine is ever upgraded it will receive a new productivity draw from $f(z)$.

Equilibrium is straightforward to define both in the cases of creative destruction and upgrading; we list the equations in Appendix G. Computation of an equilibrium is not more difficult than before since machines can be ranked along a productivity dimension that captures both age and machine-specific productivity. In this sense, the model here is isomorphic to the one without machine-specific productivity. The difference between models, of course, is quantitative: inequality in productivity across machines is now not limited to that implied by the relative price data for equipment of different vintages.

Match-Specific Productivity. Once a new match is created it has productivity e^z, with z drawn from a probability distribution with density $f(z)$ and bounded support $[0, \hat{z}]$. As before, machines age and depreciate such that the productivity at age a relative to a machine with initial productivity e^{z+gt} is $e^{z-\varphi a}$. If the machine is ever upgraded in an existing match it will maintain the same productivity draw z.

Characterization of equilibria is less straightforward under match-specific productivity, because the firm-worker pair now has two nontrivial state variables: age and productivity. Thus, whether there is scrapping/upgrading depends on the (a, z) pair. We now briefly discuss the characteristics of the stationary equilibrium in the case with creative destruction as well as in the upgrading case. The details of the analysis are to be found in Appendix G.2.

In the economy without upgrading, the question is when to scrap a machine. For every z, there is a scrapping age $\bar{a}(z)$. Because scrapping means the dissolution of the match, $\bar{a}(z)$ is an increasing function: productive matches are dissolved later.

In the economy with upgrading, matters are less transparent. There are three possible outcomes in a hypothetical (a, z) match: stay and do not upgrade, stay and upgrade, and separate. The option to separate now appears because of bad matches: if z is low enough, the machine is better employed elsewhere. For high enough values of z, however, separation never occurs, and the upgrading age $\bar{a}^U(z)$ is now decreasing in z: the better the match, the more frequent is upgrading, because vintage and productivity are complementary. Define the cutoff for this region by z_1^*. When z is low enough, i.e., below a value $z_0^* < z_1^*$, upgrading never occurs but separation occurs for an age above $\bar{a}^S(z)$. Thus, if the machine is very new, for these values of z the match remains intact and produces, but at a sufficient high age it is better for the machine to become vacant so as to find a better match, upon which upgrading could occur and the a would be reset to zero. The function $\bar{a}^S(z)$, moreover, is increasing: for a given a, an increase in z makes staying together more attractive.

Finally, when z is in an intermediate range, i.e., $z \in [z_0^*, z_1^*]$, then there are three regions: if a is low enough, the match remains intact; if it is in an intermediate range, the match separates and the machine looks for a better match; and if a is high enough, upgrading takes place. The three regions are thus defined by two cutoff functions: $\bar{a}^S(z)$, for indifference between staying and separating, and $\bar{a}^U(z)$, for indifference between separating and upgrading. Figure 4 summarizes the three regions.

6.2. Results

The one-line summary of our experiments is that neither the introduction of machine-specific nor of match-specific productivity heterogeneity changes our main results: search frictions still generate only a limited amount of wage inequality in our vintage model, and the nature of the investment decision does not affect the comparative statics in a quantitatively important way.

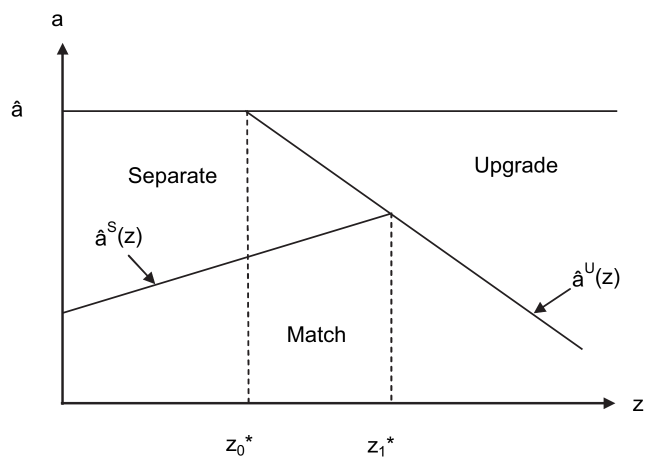

FIGURE 4. Optimal separation and upgrading decisions.

We consider the four economies implied by the combination of the two elements: (1) productivity heterogeneity (machine-specific or match-specific) and (2) means of machine replacement (creative destruction or upgrading). For each economy we assume a two-point support for productivity heterogeneity, $z \in \{z_1, z_2\}$. We normalize $z_1 = 0$ and let $z_2 > 0$ be the high-productivity machine/match. In particular, we consider economies with $z_2 \in Z_2 = [0, \bar{z}_2]$ and $f_2 \in F_2 = [0, \bar{f}_2]$. For values of $(f_2, z_2) \in F_2 \times Z_2$, we calibrate each economy to the same steady-state values of the unemployment rate, unemployment duration, replacement rate, wage income share, and average machine age as in Section 5.1.[25]

The Comparative Statics of Technological Change. Recall that in the benchmark models we have shown that (1) embodied technical change increases the unemployment rate, the unemployment duration, and wage inequality, and that it lowers the wage income share; and that (2) disembodied technical change lowers the unemployment rate, the unemployment duration, and wage inequality, and that it increases the wage income share. The models with additional heterogeneity all display the same comparative statics as the benchmark models, both qualitatively and quantitatively.[26] Figure 5 plots the comparative statics with respect to γ and ψ in the economy with match-specific heterogeneity and upgrading. The graphs for the other three economies look remarkably similar, so we omit them.

Productivity Heterogeneity and Wage Dispersion. For the case of machine-specific productivity heterogeneity, we find that both the 90–10 wage ratio and the ratio of average wages to minimum wages are very insensitive to all the values of (f_2, z_2) that we consider; recall that our one-type model is reproduced by setting either of these two variables to zero. In the model with machine-specific heterogeneity, the wages paid by high-and low-productivity machines are very close since differences in machine-specific quality are reflected in the firm's productivity and in the firm's outside value option. These two effects enter in

25. For the economies with machine-specific heterogeneity we match all steady-state targets for the range defined by $\bar{z}_2 = 0.2$ and $\bar{f}_2 = 0.5$. The calibration of economies with match-specific heterogeneity is somewhat more difficult. For the economy without upgrading we match all steady-state targets except the average age of machines on the same range of (f_2, z_2). For low values of f_2 and high values of z_2, the average age of machines increases to 13 years, which is larger than the 11.5 year target. For the economy with match-specific heterogeneity and upgrading we match most of the steady-state targets on a smaller range defined by $\bar{z}_2 = 0.2$ and $\bar{f}_2 = 0.2$. Similarly to the case without upgrading, we have some problems here in matching the average age of machines for the combination of low f_2 and high z_2 values. In addition, for high z_2 values our calibration procedure yields unemployment rates of 5% rather than the 4% target. It should also be pointed out that for this calibration exercise we are matching five variables with six parameters: we have dropped the average vacancy duration as a steady-state target. Nevertheless the calibration problem for the economy with match-specific heterogeneity is sufficiently nonlinear that we are unable to mach all steady-state variables for all (f_2, z_2) combinations.

26. The comparative statics analysis of our four economies are all taken for the same point in the interior of our set $F_2 \times Z_2$, i.e., $f_2 = 0.1$ and $z_2 = 0.1$.

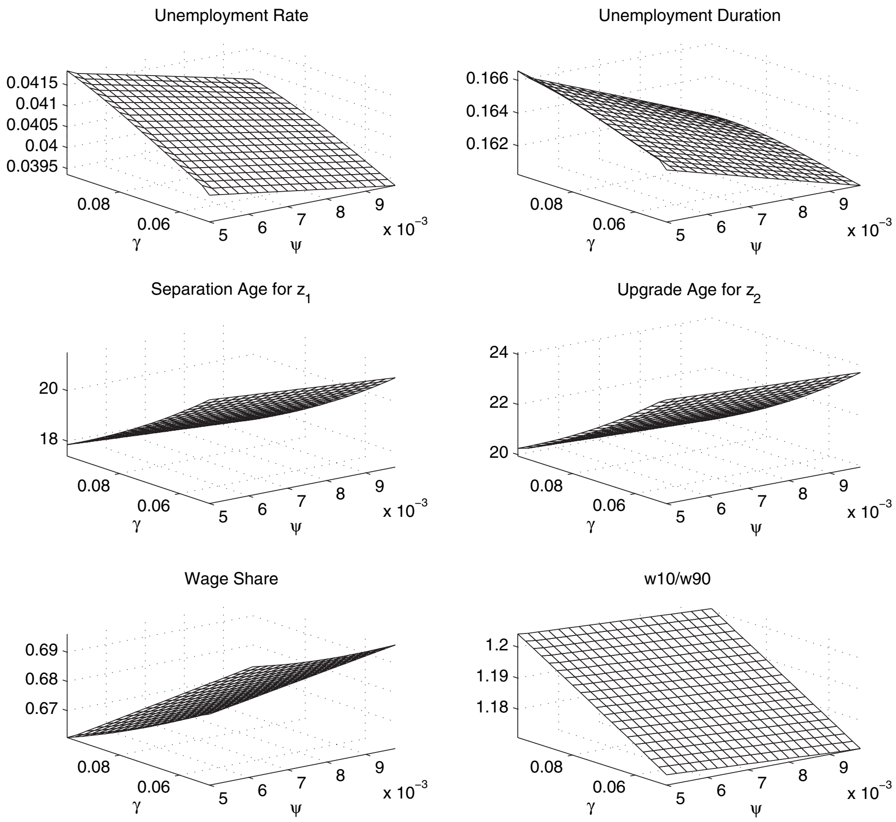

FIGURE 5. Comparative statics for embodied technical change γ and disembodied technical change ψ in the model with match-specific heterogeneity and upgrading.

the wage determination with opposite signs: high machine-productivity increases wages, but it also raises the outside option of the firm, reducing wages. Therefore, wages (and wage dispersion) are largely unaffected by this type of productivity dispersion.

Match-specific productivity-heterogeneity, on the other hand, does affect our measures of wage inequality quite substantially. This is the only new feature of our economies with heterogeneity. In the economy with creative destruction the ratio of average wages to minimum wages is essentially independent of the (f_2, z_2) values, but the 90–10 wage ratio can increase from 5% to 10% for low values of f_2 and high values of z_2. In the economy with upgrading both measures of wage inequality are affected by productivity heterogeneity. The ratio of average wages to minimum wages increases only marginally from 5% to 7%, but the 90–10 wage ratio can increase up to 20% for low values of f_2 and high values of z_2. See the last panel in Figure 5.

In this economy the surplus in a high-productivity match is substantially higher than in the low-productivity match because the outside option of machines is independent of the current match-specific productivity, so we only have a positive productivity effect on wages. The same vintage machine pays a substantially higher wage if it is in a high-productivity match. Furthermore, due to upgrading, there is indeed a substantial positive mass of new machines with high z, since upgrading occurs most frequently for good matches (see Figure 4). All this together means that the top 10% of all wage earners are essentially all in good matches with young machines. Thus an increase of the high productivity level translates almost directly into an increase in the relative wage of the top percentile group, and in the 90–10 wage ratio, whereas the average-to-minimum wage ratio is less affected.

7. Conclusions

The existing literature has pointed out that it matters qualitatively for equilibrium unemployment whether technological progress benefits only new matches or also ongoing relationships (Aghion and Howitt 1994; Mortensen and Pissarides 1998). In this paper we have shown that, if one takes the view—common in modern macroeconomics—that economic models should be calibrated and tightly parameterized to replicate certain key features of aggregate data, then the qualitative ambiguity of the growth-unemployment relationship resolves into a stark quantitative answer: the various approaches to the capital replacement problem in frictional economies all yield near-equivalent quantitative results. This conclusion is reinforced once one includes into the picture the equilibrium income shares and wage inequality, beyond the unemployment rate.

The driving force behind this result is that, quantitatively, the labor market frictions are very small: the average duration of vacancies and unemployment in the US is just 8 weeks, whereas the average life of capital is over 11 years. As a result, upgrading gives only a very small advantage compared to innovating through creative-destruction. Moreover, we explained that the fact that the calibrated model is so close to the frictionless model, quantitatively, is also responsible for the finding that equilibrium wage dispersion is tiny in the model. Only the model combining match-specific heterogeneity and upgrading generates sizeable frictional inequality. Three important caveats apply to our conclusions.

First, in economies where frictions are more severe, like continental-European labor markets where average unemployment duration can reach 6–8 months, our equivalence result could be weaker.

Second, the two replacement models may not be equivalent for the evaluation of the effects of certain labor market policies. For instance, the two approaches have different implications for employment protection policies: in a world where the introduction of more productive capital requires a reorganization of production and a firm-worker separation, employment-protection policies can have a large impact on average productivity, whereas in a world where capital can be upgraded without shedding labor, the effects of these policies will be minor.[27]

Third, throughout the analysis we maintained the hypothesis that workers are homogeneous and every source of heterogeneity that we analyzed originates from the firm's side. Random matching models with simultaneous firm and worker heterogeneity are particularly difficult to analyze because, generally, they have multiple steady states (Sattinger 1995). Thus, the right modeling strategy seems to be to introduce directed search with segmented markets. One particularly interesting dimension of workers' heterogeneity, in the context of our investigation, is age. The incentives to upgrade an existing machine are higher, the younger is the worker. Hence, creative destruction should occur mostly for older workers, whereas upgrading should be concentrated among younger workers. In line with this prediction, Bartel and Sicherman (1993) document that technical progress reduces employment for old cohorts and increases employment for younger workers. Future work should be directed toward evaluating the robustness of our finding with respect to various dimensions of workers heterogeneity.

Finally, our result has useful implications from the perspective of a recent literature that tries to identify the relative importance of disembodied technical change vis-à-vis capital-embodied productivity advances in the US by exploiting the different implications these shocks have on job creation, job destruction, and the unemployment rate (see Lopez-Salido and Michelacci 2003; Pissarides and Vallanti 2003). In our analysis, all conclusions are based on steady-state comparisons. In other words, our equivalence result holds in the long run, but we have not yet studied the short-run predictions of the different models. In this sense, we provide a cautionary remark and a suggestion: our findings here suggest that it is likely very difficult to disentangle the different sources of technical change from a low-frequency analysis of the data. However, a high-frequency analysis of the response of labor market flows to technology shocks might prove to be more informative.

Appendix A: Derivations of Typical Value Functions

The value functions of our continuous-time model can be derived as limits of a discrete time formulation. A typical derivation of the differential equations for value functions (7)–(10) goes as follows. Consider the value of a vacant firm with capital of age $a \leq \bar{a}$ at time t, $\tilde{V}(t, a)$. For a Poisson matching process, the

27. See Ljungqvist (2002) for a discussion of the impact of firing costs in a large class of environments, among others a standard matching model.

probability that the vacant firm meets a worker over a small finite time interval $[t, t+\Delta]$ is $\Delta\lambda_f$. We can define the vacancy value recursively as

$$\tilde{V}(t,a) = \Delta\lambda_f[\tilde{J}(t+\Delta, a+\Delta) - \tilde{V}(t+\Delta, a+\Delta)] + e^{-r\Delta}\tilde{V}(t+\Delta, a+\Delta),$$

where the first term is the expected capital gain from becoming a matched firm with value $\tilde{J}$ and the second term is the present value of remaining vacant at the end of the time interval. On a balanced growth path all value functions increase at the rate g over time, i.e., $\tilde{V}(t,a) = e^{gt}V(a)$ and $\tilde{J}(t,a) = e^{gt}J(a)$. Subtracting $\tilde{V}(t+\Delta, a)$ from both sides, substituting the balanced growth path expressions for $\tilde{V}$ and $\tilde{J}$, and dividing by $\Delta e^{g(t+\Delta)}$, we can rearrange the value equation into

$$\begin{aligned} -e^{-g\Delta}V(a)\frac{e^{g\Delta}-1}{\Delta} &= \lambda_f[J(a+\Delta) - V(a+\Delta)] + \frac{e^{-r\Delta}-1}{\Delta}V(a+\Delta) \\ &\quad + \frac{V(a+\Delta)-V(a)}{\Delta}. \end{aligned}$$

As we shorten the length of the time interval and take the limit for $\Delta \to 0$, we obtain the differential equation (9):

$$-gV(a) = \lambda_f[J(a) - V(a)] - rV(a) + V'(a).$$

Appendix B: Derivation of Optimal Upgrading Condition

Consider the following discretization of the investment decision when a worker–firm pair maximizes the joint value of the match.[28] The length of a time period is Δ. At $\bar{a}$ the firm and worker prefer to upgrade at $\bar{a}$ rather than delaying it by one period:

$$\begin{aligned} W(0) + J(0) - I_u \geq{} & e^{-\varphi\bar{a}}\Delta \\ & + e^{-(r-g)\Delta}\Big\{(\sigma\Delta)[V(\bar{a}+\Delta) + U] + (1-\sigma\Delta)[W(0) + J(0) - I_u]\Big\} \end{aligned}$$

The left-hand side is the joint capital value after upgrading at $\bar{a}$, and the right hand side is the flow return from production without upgrading plus the expected present value from upgrading in the next period. Note that the match separates with probability $\sigma\Delta$ and loses the upgrading opportunity. Rearranging terms and dividing by Δ we get

$$[W(0)+J(0)-I_u]\frac{1-(1-\sigma\Delta)e^{-(r-g)\Delta}}{\Delta} \geq e^{-\varphi\bar{a}} + \sigma e^{-(r-g)\Delta}[V(\bar{a}+\Delta)+U].$$

28. We consider the formulation of the problem after variables have been made stationary, that is normalized.

Taking the limit as $\Delta \to 0$ yields

$$[W(0) + J(0) - I_u](r - g + \sigma) \geq e^{-\varphi \bar{a}} + \sigma[V(\bar{a}) + U].$$

At age $(\bar{a} - \Delta)$ the firm and worker prefer not to upgrade, but to delay until $\bar{a}$:

$$W(0) + J(0) - I_u$$
$$\leq e^{-\varphi(\bar{a}-\Delta)}\Delta + e^{-(r-g)\Delta}\{(\sigma\Delta)[V(\bar{a}) + U] + (1-\sigma\Delta)[W(0) + J(0) - I_u]\}.$$

Rearranging terms and taking the limit as $\Delta \to 0$ we get

$$[W(0) + J(0) - I_u](r - g + \sigma) \leq e^{-\varphi \bar{a}} + \sigma[V(\bar{a}) + U].$$

Therefore we must have that

$$[W(0) + J(0) - I_u](r - g + \sigma) = e^{-\varphi \bar{a}} + \sigma[V(\bar{a}) + U],$$

which is equation (25) in the main text.

Appendix C: Derivation of the Surplus Function with Upgrading

In Section 4.1 we derived the differential equation for the surplus value of a matched firm–worker pair in a creative-destruction economy. This equation determines the surplus as a function of the age of the firm's machine and it is defined from the time of entry to the time of exit, $0 \leq a \leq \bar{a}$. The surplus value in an economy with upgrading satisfies the same differential equation but the terminal condition for the surplus value differs. In the creative-destruction economy, the firm/machine exits at age $\bar{a}$ and the surplus at the time of exit is zero, $S(\bar{a}) = 0$. In the economies with upgrading the machine is upgraded at age $\bar{a}$ and the surplus at that age $S(\bar{a})$ is defined in equation (29).

Substitute (31) for $(r - g)U$ into the differential equation for the surplus value (18), and solve that equation subject to the terminal condition (29) for $S(\bar{a})$,

$$\begin{aligned} S(a; \bar{a}, \lambda_f) = & \int_a^{\bar{a}} e^{-\rho_s(s-a)} \left\{ e^{-\phi s} - e^{-\phi \bar{a}} \right\} ds \\ & + [S(0) + I_0 - I_u] \Bigg\{ [r - g + \sigma(1 - \kappa_V(\bar{a}, \lambda_f))] \int_0^{\bar{a}-a} e^{-\rho_s s} ds \\ & + (1 - \kappa_V(\bar{a}, \lambda_f)) e^{-\rho_s(\bar{a}-a)} \Bigg\}, \quad \text{(C.1)} \end{aligned}$$

with $\rho_s = r - g + \sigma + (1-\beta)\lambda_f$. This is an expression for the surplus as a function of the two unknowns $(\bar{a}, \lambda_f)$ and $S(0)$. Now evaluate this expression at $a = 0$ and solve for the surplus value of a new machine $S(0)$. We obtain

$$S(0; \bar{a}, \lambda_f) = \frac{\Sigma_{1J} + (I_0 - I_u)\Sigma_{2J}}{1 - \Sigma_{2J}}, \tag{C.2}$$

with

$$\Sigma_{1J}(\bar{a}, \lambda_f) \equiv \int_0^{\bar{a}} e^{-\rho_s a} \left\{ e^{-\varphi a} - e^{-\varphi \bar{a}} \right\} da,$$

$$\Sigma_{2J}(\bar{a}, \lambda_f) \equiv \left\{ r - g + \sigma[1 - \kappa_V(\bar{a}, \lambda_f)] \right\} \int_0^{\bar{a}} e^{-\rho_s a} da + [1 - \kappa_V(\bar{a}, \lambda_f)] e^{-\rho_s \bar{a}}.$$

Appendix D: Derivation of the Steady-State Employment Dynamics

The equations describing employment dynamics are derived as follows. Consider the measure of matched vintage a firms at time t. Over a short time interval of length Δ, the approximate change in the measure is

$$\mu(t + \Delta, a) = \mu(t, a - \Delta)(1 - \Delta\sigma) + \Delta\lambda_f v(t, a - \Delta).$$

Subtracting $\mu(t, a)$ from both sides and dividing by Δ we obtain

$$\frac{\mu(t + \Delta, a) - \mu(t, a)}{\Delta} = -\frac{\mu(t, a) - \mu(t, a - \Delta)}{\Delta} - \sigma\mu(t, a - \Delta) + \lambda_f v(t, a - \Delta).$$

Taking the limit for $\Delta \to 0$ we obtain

$$\mu_t(t, a) = -\mu_a(t, a) - \sigma\mu(t, a) + \lambda_f v(t, a).$$

At steady state, these measures do not change with t, and we obtain the result stated in (33). In the economy with upgrading the initial measure of matched firms with new machines evolves according to

$$\mu(t + \Delta, 0) = \mu(t, \bar{a}) + (\Delta\lambda_f) \cdot v(t, 0) + \sum_{i=\bar{a}/\Delta}^{\hat{a}/\Delta} (\Delta\lambda_f) \cdot v(t, a_i).$$

Taking the limit for $\Delta \to 0$ we get (37) for the steady state.

Appendix E: The Invariant Employment and Vacancy Distributions

We solve the differential equation (35) for matched pairs backwards and get

$$\mu(a) = \lambda_f[\mu(0) + v(0)] \int_0^a e^{-(\sigma+\lambda_f)(a-\tilde{a})} d\tilde{a} + \mu(0) e^{-(\sigma+\lambda_f)a}, \qquad \text{(E.1)}$$

and $v(a) = v(0) + \mu(0) - \mu(a)$ for $0 \leq a \leq \bar{a}$.

In the economy without upgrading $\mu(0) = 0$ and we get closed-form expressions for the employment and vacancy densities

$$\mu(a)/\mu = \frac{1 - e^{-(\sigma+\lambda_f)a}}{\bar{a} - \frac{1}{\sigma+\lambda_f}(1 - e^{-(\sigma+\lambda_f)\bar{a}})}, \qquad \text{(E.2)}$$

$$v(a)/v = \frac{\sigma + \lambda_f e^{-(\sigma+\lambda_f)a}}{\bar{a}\sigma + \frac{\lambda_f}{\sigma+\lambda_f}(1 - e^{-(\sigma+\lambda_f)\bar{a}})}. \qquad \text{(E.3)}$$

In the economy with upgrading we have to solve for the employment density of new machines $\mu(0)$. We solve the differential equation (38) for vacancies on the interval $[\bar{a}, \hat{a}]$ backwards and get

$$v(a) = e^{-\lambda_f(a-\bar{a})} v(\bar{a}). \qquad \text{(E.4)}$$

The total measure of vacancies on $[\bar{a}, \hat{a}]$ is then

$$\int_{\bar{a}}^{\hat{a}} v(a) da = v(\bar{a}) A_2 \text{ with } A_2 \equiv \int_0^{\hat{a}-\bar{a}} e^{-\lambda_f a} da. \qquad \text{(E.5)}$$

We evaluate expression (E.1) at $\bar{a}$ to get the measure of existing matches that upgrade:

$$\mu(\bar{a}) = [\mu(0) + v(0)]\lambda_f A_1 + \mu(0) e^{-(\sigma+\lambda_f)\bar{a}} \quad \text{with} \quad A_1 \equiv \int_0^{\bar{a}} e^{-(\sigma+\lambda_f)a} da. \qquad \text{(E.6)}$$

We substitute (E.6) and (E.5) into (37) and get

$$\begin{aligned} \mu(0) &= \mu(\bar{a}) + \lambda_f A_2 v(\bar{a}) \\ &= \mu(\bar{a}) + \lambda_f A_2[\mu(0) + v(0) - \mu(\bar{a})] \\ &= [1 - \lambda_f A_2][\{\mu(0) + v(0)\}\lambda_f A_1 + \mu(0) e^{-(\sigma+\lambda_f)\bar{a}}] \\ &\quad + \lambda_f A_2[\mu(0) + v(0)]. \end{aligned}$$

We can solve this expression for the density of new employed machines as a function of new vacant machines

$$\mu(0) = Bv(0), \text{ with} \qquad B = \frac{(1-\lambda_f A_2)\lambda_f A_1 + \lambda_f A_2}{1-(1-\lambda_f A_2)(\lambda_f A_1 + e^{-(\sigma+\lambda_f)\bar{a}}) - \lambda_f A_2}. \tag{E.7}$$

Note that B can be simplified to

$$B = \frac{1 - e^{-\lambda_f(\hat{a}-\bar{a})}\{1-[1-e^{-(\sigma+\lambda_f)\bar{a}}]\lambda_f/(\sigma+\lambda_f)\}}{e^{-\lambda_f(\hat{a}-\bar{a})}\left[1-e^{-(\sigma+\lambda_f)\bar{a}}\right]\sigma/(\sigma+\lambda_f)}.$$

For the calibration of our economy, B is very large since the denominator is close to zero. This will be important when we obtain numerical solutions of the steady state.

Substituting (E.7) and (37) into the expression for the density of employed machines at the upgrade age $\bar{a}$, (E.6), yields $\mu(\bar{a}) = v(0)(1+B)\lambda_f A_1 + Bv(0)e^{-(\sigma+\lambda_f)\bar{a}}$ or

$$\mu(\bar{a}) = C_1 v(0), \text{ with} \qquad C_1 = (1+B)\lambda_f A_1 + Be^{-(\sigma+\lambda_f)\bar{a}}. \tag{E.8}$$

Evaluating (37) at $\bar{a}$ and solving for $v(\bar{a})$ we have $v(\bar{a}) = \mu(0) + v(0) - \mu(\bar{a})$. After we substitute (E.8) for $\mu(\bar{a})$ and (E.7) for $\mu(0)$ we have

$$v(\bar{a}) = C_2 v(0), \text{ with} \qquad C_2 \equiv (1+B)(1-\lambda_f A_1) - Be^{-(\sigma+\lambda_f)\bar{a}}. \tag{E.9}$$

Integrating the employment density (E.1) over the interval $[0, \bar{a}]$ yields total employment

$$\int_0^{\bar{a}} \mu(a)da = \lambda_f[\mu(0)+v(0)]\int_0^{\bar{a}}\left[\int_0^a e^{-(\sigma+\lambda_f)\tilde{a}}d\tilde{a}\right]da + \mu(0)\int_0^{\bar{a}} e^{-(\sigma+\lambda_f)\tilde{a}}d\tilde{a}.$$

Substituting (E.7) for $\mu(0)$ in (37) yields

$$\int_0^{\bar{a}} \mu(a)da = C_3 v(0), \text{ with} \tag{E.10}$$

$$C_3 = (1+B)\lambda_f(\bar{a}-A_3)/(\sigma+\lambda_f) + BA_3$$

$$A_3 = \int_0^{\bar{a}} e^{-(\sigma+\lambda_f)a}da.$$

We can now calculate the total measure of vacancies on the interval $[0, \bar{a}]$. Using (37) we get

$$\int_0^{\bar{a}} v(a)da = \int_0^{\bar{a}} [\mu(0) + v(0) - \mu(a)]da = [\mu(0) + v(0)]\bar{a} - \int_0^{\bar{a}} \mu(a)da,$$

and using equations (E.7) and (E.10) we get

$$\int_0^{\bar{a}} v(a)da = C_4 v(0), \text{ with} \tag{E.11}$$
$$C_4 = (1 + B)[\bar{a} - (\bar{a} - A_3)\lambda_f/(\sigma + \lambda_f)] - BA_3.$$

Combining equations (E.5), (E.9), and (E.11) yields total vacancies as

$$\int_0^{\hat{a}} v(a)da = C_5 v(0), \text{ with} \tag{E.12}$$
$$\begin{aligned} C_5 \equiv {} & (1 + B)[\bar{a} - (\bar{a} - A_3)\lambda_f/(\sigma + \lambda_f) + A_2(1 - \lambda_f A_1)] \\ & - B\big[A_3 + A_2 e^{-(\sigma+\lambda_f)\bar{a}}\big]. \end{aligned}$$

To get the density of new firms coming into the economy with new machines we use the definition of labor market tightness

$$\theta = \frac{\int_0^{\hat{a}} v(a)da}{1 - \int_0^{\bar{a}} \mu(a)da} = \frac{C_5 v(0)}{1 - C_3 v(0)}$$

and solve for $v(0)$

$$v(0) = \frac{\theta}{\theta C_3 + C_5}.$$

Note that both C_3 and C_5 are linear in B, and since B is large for the calibration of the economy, entry is essentially zero. A good approximation of the employment and vacancy densities is then obtained by multiplying $v(0)$ with B and dividing all densities with B, or

$$\begin{aligned} \tilde{v}(0) &= Bv(0) = \theta/[\theta \tilde{C}_3 + \tilde{C}_5], \text{ with} \\ \tilde{C}_5 &= (1 + 1/B)[\bar{a} - (\bar{a} - A_3)\lambda_f/(\sigma + \lambda_f) + A_2(1 - \lambda_f A_1)] \\ &\quad - [A_3 + A_2 e^{-(\sigma+\lambda_f)\bar{a}}] \\ &\approx [\bar{a} - (\bar{a} - A_3)\lambda_f/(\sigma + \lambda_f) + A_2(1 - \lambda_f A_1)] - \big[A_3 + A_2 e^{-(\sigma+\lambda_f)\bar{a}}\big] \\ \tilde{C}_4 &= (1 + 1/B)\lambda_f(\bar{a} - A_3)/(\sigma + \lambda_f) + A_3 \\ &\approx \lambda_f(\bar{a} - A_3)/(\sigma + \lambda_f) + A_3 \end{aligned}$$

$$\begin{aligned}
\tilde{C}_3 &= (1+1/B)\lambda_f(\bar{a}-A_3)/(\sigma+\lambda_f)+A_3\\
&\approx \lambda_f(\bar{a}-A_3)/(\sigma+\lambda_f)+A_3\\
\tilde{C}_2 &= (1+1/B)(1-\lambda_f A_1)-e^{-(\sigma+\lambda_f)\bar{a}}\\
&\approx (1-\lambda_f A_1)-e^{-(\sigma+\lambda_f)\bar{a}}\\
\tilde{C}_1 &= (1+1/B)\lambda_f A_1+e^{-(\sigma+\lambda_f)\bar{a}}\\
&\approx \lambda_f A_1+e^{-(\sigma+\lambda_f)\bar{a}}\\
\mu(0) &= \tilde{v}(0),\ \mu(\bar{a})=\tilde{C}_1\tilde{v}(0),\ v(\bar{a})=\tilde{C}_2\tilde{v}(0).
\end{aligned}$$

Appendix F: Proof of Proposition 1

We model the disappearance of the matching friction by letting the shift parameter of the matching function $\kappa \to \infty$. Given that all the relevant equations are written in terms of the endogenous variable λ_f, which is increasing in κ, our line of proof will be based on taking limits as $\lambda_f \to \infty$. We use the key equilibrium conditions of the two replacement models (creative-destruction and upgrading) to show that (i) the economy with upgrading converges to the economy with creative destruction and that (ii) the latter converges to the frictionless economy when the instantaneous meeting rate for firms becomes large enough and $I_0 = I_u$.

Precisely, we first show that as $\lambda_f \to \infty$ the "extra" terms that appear in the conditions (JCU) and (JU), but do not appear in the conditions (JCD) and (JD), vanish. Second, we show that the expressions for the surplus function converge as well. Third, we show that the (JCD) condition converges to the frictionless free-entry condition (4) and that the wage function $w(a)$ implicitly defined in (14) collapses to the marginal product of labor $e^{-\varphi\bar{a}}$, i.e., the unique competitive wage. Finally, we show that the distribution of employed machines in the two economies converge to the competitive equilibrium distribution.

Proof. Consider the extra term in (JCU) and let $\lambda_f \to \infty$. The expression $\kappa_V(\bar{a}, \lambda_f)$ converges to 1 and $S(0; \bar{a}, \lambda_f)$ converges to zero. The latter limit is clear from simple inspection of (C.2) in Appendix 7, since both $\Sigma_{1J}(\bar{a}, \lambda_f)$ and $\Sigma_{2J}(\bar{a}, \lambda_f)$ converge to zero as λ_f gets large. Hence, the extra term in (JCU) converges to zero. Consider now the extra term in (JU) and let $\lambda_f \to \infty$. Since $\kappa_V(a, \lambda_f)$ converges to 1 for all a's, $S(0; \bar{a}, \lambda_f)$ converges to zero, and $I_0 = I_u$, then this term goes to zero as well.

It is easy to see, from (C.1) in Appendix C, that the extra term (the second and third lines) in the surplus function of the economy with upgrading goes to zero as $\lambda_f \to \infty$, and thus the expressions for the surplus in the two economies converge.

Now consider how equation (JCD) changes as $\lambda_f \to \infty$. Using the surplus expression (21) in (JCD) and integrating the right-hand side, yields

$$I_0 = \frac{(1-\beta)\lambda_f}{\rho_2}\left\{\frac{1-e^{-(r-g+\varphi)\bar{a}}}{r-g+\varphi} - \frac{\rho_2}{\rho_1}e^{-\varphi\bar{a}}\frac{1-e^{-(r-\varphi)\bar{a}}}{r-\varphi} \right.$$
$$\left. + e^{-\rho_2\bar{a}}\frac{1-e^{g\bar{a}}}{\sigma+(1-\beta)\lambda_f}\frac{\varphi}{\rho_1}\right\}, \quad \text{(F.1)}$$

where we have introduced the notation $\rho_0 = r - g + \sigma$, $\rho_1 = \rho_0 + (1-\beta)\lambda_f$, and $\rho_2 = \rho_1 + \varphi$. Taking the limit of expression (F.1) as $\lambda_f \to \infty$, we get

$$I_0 = \frac{1-e^{-(r-g+\varphi)\bar{a}}}{r-g+\varphi} - e^{-\varphi\bar{a}_{\min}}\frac{1-e^{-(r-g)\bar{a}}}{r-g}$$
$$= \int_0^{\bar{a}} e^{-(r-g+\varphi)a}\left[1-e^{-\varphi(\bar{a}-a)}\right]da,$$

which is the key equilibrium condition (4) of the frictionless model in Section 2. Now, consider equation (18) that implicitly defines the surplus function. As $\lambda_f \to \infty$, it is easy to see that the term $\lambda_f(1-\beta)S(a;\bar{a},\lambda_f)$ converges to $\left(e^{-\varphi a} - e^{-\varphi\bar{a}}\right)$. Using this result and the equilibrium condition $(r-g)U = e^{-\varphi\bar{a}}$, we notice immediately that equation (18) implies that $(r-g+\sigma)S(a) - S'(a) = 0$. Using this result in the wage equation (14), we obtain that $w(a) = e^{-\varphi\bar{a}}$ for every a, which is the competitive wage.

It only remains to show that the vacancy and employment distributions converge, but this is trivial once it is recognized that as the meeting rate for firms goes to infinity, the measure of vacancies tends to zero and the employment density is simply $\mu(a)/\mu = 1/\bar{a}$ like in the frictionless economy. □

Appendix G: Equilibrium with Machine-Specific Productivity Differences

G.1. The Economy without Upgrading

Optimal Entry and Exit. Capital values are functions of a machine's age a and quality z. Machines age, but the quality of a machine does not change over time.

$$(r-g)J(a,z) = \max\Big\{e^{z-\varphi a} - w(a,z) - \sigma[J(a,z) - V(a,z)] + J_a(a,z),$$
$$(r-g)V(a,z)\Big\}, \quad \text{(G.1)}$$

$$(r-g)W(a,z) = \max\{w(a,z) - \sigma[W(a,z) - U] + W_a(a,z), (r-g)U\}, \quad \text{(G.2)}$$

$$(r-g)V(a,z) = \max\{\lambda_f[J(a,z) - V(a,z)] + V_a(a,z), 0\}, \tag{G.3}$$

$$(r-g)U = b + \int_0^{\hat{z}} \int_0^{\hat{a}} \lambda_w(a,z)[W(a,z) - U]\, da\, dz. \tag{G.4}$$

The surplus value of a match is

$$S(a,z) = [J(a,z) - V(a,z)] + [W(a,z) - U],$$

and the worker receives a share β of the surplus:

$$W(a,z) - U = \beta S(a,z) \text{ and } J(a,z) - V(a,z) = (1-\beta)S(a,z).$$

The implied differential equation for the surplus value is then

$$[r - g + \sigma + \lambda_f(1-\beta)]S(a,z) = e^{z-\varphi a} - (r-g)U + S_a(a,z). \tag{G.5}$$

This differential equation can be solved conditionally on the terminal condition that defines the optimal time of exit $\bar{a}(z)$ (the job-destruction condition)

$$S[\bar{a}(z), z] = S_a[\bar{a}(z), z] = 0. \tag{G.6}$$

This implies that

$$e^{z-\varphi\bar{a}} = (r-g)U \text{ or } z - \varphi\bar{a}(z) = \kappa = \log[(r-g)U], \tag{G.7}$$

and the surplus capital value is

$$\begin{aligned} S(a,z) &= \int_a^{\bar{a}(z)} e^{-[r-g+\sigma+(1-\beta)\lambda_f](\tilde{a}-a)} \left[e^{z-\varphi\tilde{a}} - (r-g)U\right] d\tilde{a} \\ &= \int_0^{[z-\phi a-\kappa]/\phi} e^{-[r-g+\sigma+(1-\beta)\lambda_f]\tilde{a}} \left[e^{(z-\varphi a)-\varphi\tilde{a}} - (r-g)U\right] d\tilde{a} \\ &= S(z-\phi a). \end{aligned}$$

The vacancy value is

$$\begin{aligned} V(a,z) &= \lambda_f(1-\beta) \int_a^{\bar{a}(z)} e^{-(r-g)(\tilde{a}-a)} S(z-\varphi\tilde{a}) d\tilde{a} \\ &= \lambda_f(1-\beta) \int_0^{[z-\varphi a-\kappa]/\phi} e^{-(r-g)\tilde{a}} S(z-\varphi a-\varphi\tilde{a}) d\tilde{a} \\ &= V(z-\varphi a). \end{aligned}$$

The free-entry condition reads

$$I_0 = \int_0^{\hat{z}} f(z)V(z)dz.$$

Employment and Vacancy Measures. We have

$$v_a(a,z) = \sigma\mu(a,z) - \lambda_f v(a,z), \text{ for } 0 \le a \le \bar{a}(z), \tag{G.8}$$

$$\mu_a(a,z) = \lambda_f v(a,z) - \sigma\mu(a,z), \text{ for } 0 \le a \le \bar{a}(z). \tag{G.9}$$

The evolution of matched machines is the mirror image of the evolution of vacancies, i.e., $\mu_a(a,z) = -v_a(a,z)$. This implies that the number of type z vacant and matched machines of age $a < \bar{a}(z)$ remains constant:

$$v(a,z) + \mu(a,z) = v(0,z) + \mu(0,z), \text{ for } 0 \le a \le \bar{a}(z). \tag{G.10}$$

Because all firms proceed first to the search pool with their new machines, $\mu(0,z) = 0$. For $a \in [0, \bar{a}(z)]$, the evolution of $\mu(a,z)$ therefore follows

$$\mu_a(a,z) = \lambda_f f(z) e_f - (\sigma + \lambda_f)\mu(a,z).$$

The solution of the differential equations yields

$$\mu(a,z) = f(z) e_f \lambda_f \int_0^a e^{-(\sigma+\lambda_f)\tilde{a}} d\tilde{a},$$

$$v(a,z) = f(z) e_f \{1 - \lambda_f \int_0^a e^{-(\sigma+\lambda_f)\tilde{a}} d\tilde{a}\}$$

for $0 \le a \le \bar{a}(z)$. Given the vacancy distribution $v(a,z)$ the rate at which workers meet machines is $\lambda_w(a,z) = \lambda_w v(a,z) / \int v(a,\tilde{z}) dz$ for $0 \le a \le \bar{a}(z)$ and zero otherwise.

G.2. The Economy with Upgrading

Optimal Entry and Upgrading. The definitions of the capital values for a matched machine and worker are the same as without upgrading, equations (G.1) and (G.2). The capital values of a vacancy and an unemployed worker are

$$(r-g)V(a,z) = \max\left\{\lambda_f\left[\max\left\{J(a,z), E_{\tilde{z}}[J(0,\tilde{z})] - I_u^J(a,z)\right\} - V(a,z)\right] + V_a(a,z), 0\right\}, \tag{G.11}$$

$$(r-g)U = b + \int_0^{\hat{z}}\int_0^{\hat{a}} \lambda_w(a,z)\left[\max\left\{W(a,z), E_{\tilde{z}}[W(0,\tilde{z})] - I_u^W(a,z)\right\} - U\right] da\, dz, \tag{G.12}$$

where $E_{\tilde{z}}$ denotes the expectation with respect to the density $f(\tilde{z})$. The gains from upgrading in an existing match are

$$\begin{aligned} G(a,z) &= E_{\tilde{z}}[J(0,\tilde{z}) + W(0,\tilde{z})] - I_u - J(a,z) - W(a,z) \\ &= E_{\tilde{z}}[S(0,\tilde{z})] + I_0 - I_u - S(a,z) - V(a,z), \end{aligned}$$

using the surplus-sharing rule and the free-entry condition. An existing match will upgrade the machine as soon as the gains from upgrading are nonnegative and there are no gains from a marginal delay of the upgrading decision:

$$G[\bar{a}(z), z] = G_a[\bar{a}(z), z] = 0.$$

These two conditions imply

$$E_{\tilde{z}}[S(0,\tilde{z})] + I_0 - I_u = S[\bar{a}(z), z] + V[\bar{a}(z), z], \tag{G.13}$$

$$(r - g + \sigma)S[\bar{a}(z), z] = e^{z - \varphi\bar{a}(z)} - (r - g)\{V[\bar{a}(z), z] + U\}. \tag{G.14}$$

The differential equation for the surplus of existing matches, $S(a, z)$ for $a \leq \bar{a}(z)$, is defined as in (G.5). There are no existing matches with machines of age $a \geq \bar{a}(z)$. When a previously vacant type z machine of age $a > \bar{a}(z)$ meets an unemployed worker the gains from upgrading are

$$\begin{aligned} G^m(a,z) &= E_{\tilde{z}}[J(0,\tilde{z}) + W(0,\tilde{z})] - I_u - V(a,z) - U \\ &= E_{\tilde{z}}[S(0,\tilde{z})] + I_0 - I_u - V(a,z). \end{aligned}$$

Incorporating the optimal upgrading decision, the differential equation for the vacancy value function becomes

$$(r - g)V(a,z) - V_a(a,z) = \begin{cases} \lambda_f(1-\beta)S(a,z) & \text{for } a < \hat{a}(z), \\ \lambda_f(1-\beta)G^m(a,z) & \text{for } a \geq \hat{a}(z). \end{cases}$$

Using the surplus-sharing rule the flow value of unemployment (G.12) now becomes

$$\begin{aligned} (r-g)U = b + \beta \int_0^{\hat{z}} \Bigg[\int_0^{\bar{a}(z)} & \lambda_w(a,z) S(a,z) da \\ & + \int_{\bar{a}(z)}^{\hat{a}} \lambda_w(a,z) G^m(a,z)\, da \Bigg] f(z)\, dz. \end{aligned}$$

Employment and Vacancy Measures. Define the total measure of machines that are upgrading at a point in time as

$$\mu(0) = \int_0^{\hat{z}} \left\{ \mu[\bar{a}(z), z] + \lambda_f \int_{\bar{a}(z)}^{\hat{a}} v(a, z) da \right\} dz.$$

This is also the measure of all new machines. The distribution over new machines according to type z is then

$$\mu(0, z) = f(z)\mu(0) \text{ and } v(0, z) = f(z)e_f.$$

For type z machines the employment distribution for $a \leq \bar{a}(z)$ is described in the same way as in the case of no upgrading, equations (G.8), (G.9), and (G.10). Vacant machines that are older than the critical upgrading age but younger than the critical age when they can no longer be upgraded, $\bar{a}(z) \leq a \leq \hat{a}$, stay in the vacancy pool until they find a worker and upgrade

$$v_a(a, z) = -\lambda_f v(a, z). \tag{G.15}$$

To solve for the employment and vacancy distributions we proceed the same way as in the upgrading case without machine-specific productivity heterogeneity.

Appendix H: Equilibrium with Match-Specific Productivity Differences

H.1. The Economy without Upgrading

Optimal Entry and Exit. The definitions of the capital values of employed and unemployed workers are the same as in the economy with machine-specific heterogeneity and no upgrading, equations (G.2) and (G.4). The capital value equations for a matched and a vacant machine are

$$(r-g)J(a, z) = \max\Big\{ e^{z-\varphi a} - w(a, z) - \sigma[J(a, z) - V(a)] + J_a(a, z), \\ (r-g)V(a) \Big\}, \tag{H.1}$$

$$(r-g)V(a) = \max\left\{ \lambda_f E_{\tilde{z}}[J(a, \tilde{z}) - V(a)] + V_a(a), 0 \right\}. \tag{H.2}$$

The differential equation for the surplus value is

$$\{r-g+\sigma\}S(a, z) = e^{z-\varphi a} - (r-g)U - \lambda_f(1-\beta)E_{\tilde{z}}[S(a, \tilde{z})] + S_a(a, z). \tag{H.3}$$

Since productivity z is match-specific and random, the flow return on the outside option of a machine vacancy is now with respect to the expected surplus, cf.

equation (G.5). Optimal separation of a match depends on the match-specific productivity, and at the separation age $\bar{a}(z)$ the surplus is zero and there are no gains from a marginal delay of separation, equation (G.6). It is apparent that the flow return on surplus is increasing in the match-specific productivity. Therefore, more productive matches will separate later, i.e., $\bar{a}(z)$ is increasing.

The job-destruction condition is implied by the zero gain from a marginal delay of separation:

$$e^{z-\varphi\bar{a}(z)} = \lambda_f(1-\beta)E_{\tilde{z}}\{S[\bar{a}(z), \tilde{z}]\} + (r-g)U. \tag{H.4}$$

Different from the economy with machine-specific productivity-heterogeneity there may now be a positive flow return on the outside option of a vacancy since the machine might draw a higher match productivity. This chance of finding a match with a higher productivity increases the cost of staying in a match with given productivity. Machines in a match of quality z separate at age $\bar{a}(z)$. A vacant firm stays in the vacancy pool until it reaches the maximal age for active matches $\bar{a} = \max_z \bar{a}(z)$. Given the zero terminal value of the surplus at the separation age we can solve the surplus differential equation forward for the surplus capital value

$$\begin{aligned} S(a,z) = \int_a^{\bar{a}(z)} & e^{-(r-g+\sigma)(\tilde{a}-a)} \\ & \times \left\{ e^{z-\phi\tilde{a}} - \lambda_f(1-\beta)E_{\tilde{z}}[S(\tilde{a},\tilde{z})] - (r-g)U \right\} d\tilde{a}. \end{aligned}$$

Since all machines exit at age $\bar{a}$, the terminal vacancy value is zero, $V(\bar{a}) = 0$. Using the surplus-sharing rule we can solve the differential equation (H.2) for the vacancy value forward and conditionally on the terminal value obtain the capital value of a vacancy,

$$V(a) = \lambda_f(1-\beta)\int_a^{\bar{a}} e^{-(r-g)(\tilde{a}-a)} E_{\tilde{z}}[S(\tilde{a},\tilde{z})]d\tilde{a}.$$

The free-entry condition for new machines is $I_0 = V(0)$.

Surplus and Vacancy Functions with a Finite Set of z Types. Using the fact that $\bar{a}_i < \bar{a}_{i+1}$ we can define a sequence of differential equation systems for the surplus function $\{S_i(a) = S(a, z_i)\}$. On the interval $\bar{a}_{i-1} \le a \le \bar{a}_i$ the system defines the surplus functions $S_j(a)$, for $j = i, \dots, Z$, through

$$\begin{aligned} S_j'(a) = {} & [r-g+\sigma+\lambda_f(1-\beta)f_j]S_j(a) \\ & + \lambda_f(1-\beta)\sum_{k>j} f_k S_k(a) - e^{z_j-\varphi a} + (r-g)U. \end{aligned}$$

These systems can be solved sequentially working backwards using the terminal conditions $S_i(\bar{a}_i) = 0$. Conditional on the piecewise-defined surplus functions, we can then solve the differential equation for the vacancy value recursively

$$(r-g)V(a) = \lambda_f(1-\beta)\sum_{j\geq i} f_j S_j(a) + V_a(a) \text{ on } [\bar{a}_{i-1}, \bar{a}_i].$$

Employment and Vacancy Measures with Finite z Types. Without loss of generality assume that all matches operate at least for some time, that is $\bar{a}_1 = \bar{a}(z_1) > 0$. The total measure of young machines with age $a \leq a_1$ evolves according to

$$\begin{aligned} \mu(0) &= 0 \quad \text{and} \quad v(0) = e_f, \\ \mu'(a) &= \lambda_f v(a) - \sigma\mu(a), \\ v'(a) &= \sigma\mu(a) - \lambda_f v(a). \end{aligned}$$

Thus for $a \leq \bar{a}_1$

$$v(a) + \mu(a) = v(0) + \mu(0).$$

The measure of match type i is then given by

$$\mu_i(a) = f_i\mu(a).$$

In general, the distribution evolves according to

$$\begin{aligned} \mu(\bar{a}_i+) &= \mu(\bar{a}_i-)\left[\sum_{j>i} f_j\right]\Big/\left[\sum_{j>i-1} f_j\right], \\ v(\bar{a}_i+) &= v(\bar{a}_i-) + f_i\mu(\bar{a}_i-), \\ v'(a) &= \sigma\mu(a) - \lambda_f v(a)\sum_{j>i} f_j \text{ for } \bar{a}_i \leq a \leq \bar{a}_{i+1}, \\ \mu'(a) &= -\sigma\mu(a) + \lambda_f v(a)\sum_{j>i} f_j \text{ for } \bar{a}_i \leq a \leq \bar{a}_{i+1}, \\ \mu_j(a) &= \begin{cases} f_{ji}\mu(a) & \text{for } j > i \text{ with } f_{ji} = f_j \Big/ \sum_{s>i} f_s, \\ 0 & \text{for } j \leq i, \end{cases} \end{aligned}$$

where $g(\bar{a}+)$ denotes the right-hand side limit of the function g at $\bar{a}$, $g(\bar{a}+) \equiv \lim_{\varepsilon>0,\varepsilon\to 0} g(\bar{a}+\varepsilon)$. The employment distributions are continuous functions of age and the total employment and vacancy functions are piecewise continuous with discontinuities at the critical exit ages. Again we normalize the employment and vacancy distribution by dividing through with the entry rate of new machines.

H.2. The Economy with Upgrading

Optimal Entry, Separation, and Upgrading. The definitions of the capital values for vacancy and an unemployed worker are

$$(r-g)V(a) = \max\left\{\lambda_f\left\{E_{\tilde{z}}\left[\max\{J(0,\tilde{z}) - I_u^J(a,\tilde{z}), J(a,\tilde{z})\}\right] - V(a)\right\}\right.$$

$$\left. + V_a(a), 0\right\} \tag{H.5}$$

$$(r-g)U = b + \int_0^{\hat{a}} \lambda_w(a)\left\{E_{\tilde{z}}\left[\max\left\{W(0,\tilde{z}) - I_u^W(a,\tilde{z}), W(a,\tilde{z})\right\}\right] - U\right\} da \tag{H.6}$$

and the capital value definitions of a matched machine and worker are the same as without upgrading, equations (H.1) and (G.2). We get the differential equation for the surplus value from the surplus definition and the expression for the capital value of a matched worker and machine

$$(r-g+\sigma)S(a,z) + (r-g)[V(a)+U] = e^{z-\varphi a} + S_a(a,z) + V_a(a,z). \tag{H.7}$$

Note that we can no longer eliminate the vacancy value from the surplus expression, rather we have to solve the differential equation system for the surplus and the vacancy value jointly. To this end, define the gains from upgrading in a match with an age a machine and type z productivity

$$G(a,z) = J(0,z) + W(0,z) - I_u - J(a,z) - W(a,z).$$

We distinguish between the separation age of a match $\bar{a}^S(z)$ and the upgrading age of a match $\bar{a}^U(z)$, $\bar{a}^S(z) \leq \bar{a}^U(z)$. If $\bar{a}^S(z) = \bar{a}^U(z)$ then existing matches do not separate. Using the surplus-sharing rules, the expression for the gains from upgrading simplifies to

$$G(a,z) = \begin{cases} S(0,z) + I_0 - I_u - V(a) - S(a,z) & \text{for } a \leq \bar{a}^S(z) \\ S(0,z) + I_0 - I_u - V(a) & \text{for } a > \bar{a}^S(z). \end{cases}$$

Furthermore, the gains from a marginal delay of upgrading in an existing match, $a \leq \bar{a}^S(z)$, are

$$G_a(a,z) = e^{z-\varphi a} - (r-g)[U + V(a)] - (r-g+\sigma)S(a,z).$$

We can use the surplus-sharing rule for existing matches and the gains from upgrading in the expression for the vacancy value and obtain

$$(r-g)V(a) = (1-\beta)\lambda_f E_{\tilde{z}}\left[\max\{S(0,\tilde{z}) + I_0 - I_u - V(a), S(a,\tilde{z})\}\right] + V_a(a). \tag{H.8}$$

Equations (H.7) and (H.8) define a system of differential equations in $S(a, z)$ and $V(a)$, which has to be solved jointly.

For a finite number of productivity types z the optimal separation/upgrading decisions are characterized by the critical age values $\{\bar{a}_i^S, \bar{a}_i^U : i = 1, \ldots, Z\}$ such that

$$\bar{a}_i^S < \bar{a}_i^U : S_i(\bar{a}_i^S) = S_i'(\bar{a}_i^S) = 0, G_i(\bar{a}_i^S) < 0, G_i(\bar{a}_i^U) = G_i'(\bar{a}_i^U) = 0$$
$$\bar{a}_i^S = \bar{a}_i^U : S_i(\bar{a}_i^S) \geq 0, G_i(\bar{a}_i^U) = G_i'(\bar{a}_i^U) = 0.$$

Employment and Vacancy Measures. Define the total measure of type z_i matches that are upgrading at a point in time

$$\mu_i(0) = \begin{cases} \mu_i(\bar{a}_i^U) + f_i \lambda_f \int_{\bar{a}_i^U}^{\hat{a}} v(a) da & \text{for } \bar{a}_i^S = \bar{a}_i^U, \\ f_i \lambda_f \int_{\bar{a}_i^U}^{\hat{a}} v(a) da & \text{for } \bar{a}_i^S < \bar{a}_i^U. \end{cases}$$

The employment distributions evolve according to

$$\mu_i'(a) = f_i \lambda_f v(a) - \sigma \mu_i(a) \text{ for } 0 \leq a \leq \bar{a}_i^S,$$
$$\mu_i(a) = 0 \text{ for } \bar{a}_i < a \leq \hat{a}.$$

The vacancy distribution evolves according to

$$v'(a) = \sigma \sum_i \mu_i(a) - \lambda_f v(a) \sum_i \chi(a \leq \bar{a}_i^S \text{ or } a \geq \bar{a}_i^U) f_i$$

and

$$v(\bar{a}_i^S +) = v(\bar{a}_i^S) + \mu_i(\bar{a}_i^S) \text{ if } \bar{a}_i^S < \bar{a}_i^U,$$

subject to the initial condition $v(0) = e_f$.

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